

Calculus and Analytic Geometry Math 232, Section 5

First Test, October 12, 2005

1. Find the area of the region bounded by the curves $y = x^3$, $y = 3x - 2$, $y = 0$ in the first quadrant.

Solution: The intersection points are at $x = 1$ and $x = -2$. Since the region should be in the first quadrant, the region is bounded above by x^3 for $0 \leq x \leq 1$ and bounded below by $y = 0$ for $0 \leq x \leq 2/3$ and $3x - 2$ for $2/3 \leq x \leq 1$. Therefore,

$$A = \int_0^{2/3} x^3 dx + \int_{2/3}^1 (x^3 - (3x - 2)) dx = \int_0^1 x^3 dx - \int_{2/3}^1 (3x - 2) dx = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

2. Find the volume of the solid generated by revolving the region R about the x -axis, where R is bounded by $y = x^2$ and $y = 2 - x$.

Solution: The intersection points are at $x = -2$ and $x = 1$. Therefore, by the washer method,

$$\begin{aligned} V &= \pi \int_{-2}^1 ((2 - x)^2 - x^4) dx = \pi \int_{-2}^1 (4 - 4x + x^2 - x^4) dx \\ &= \pi \left[4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_{-2}^1 = \frac{72}{5}\pi. \end{aligned}$$

3. Find the volume of the solid generated by revolving the region R about the y -axis, where R is bounded by $y = x(1 - x)$, $y = 3x(1 - x)$.

Solution: The intersection points are at $x = 0$ and $x = 1$. Therefore, by the shell method,

$$\begin{aligned} V &= 2\pi \int_0^1 x(3x(1 - x) - x(1 - x)) dx \\ &= 4\pi \int_0^1 (x^2 - x^3) dx \\ &= 4\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = 4\pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\pi}{3}. \end{aligned}$$

4. Find the average value of the function $f(x) = \frac{x^2}{1 + x^3}$ on $[0, 3]$.

Solution: Using the substitution $u = 1 + x^3$ we obtain

$$f_{ave} = \frac{1}{3} \int_0^3 \frac{x^2}{1+x^3} dx = \frac{1}{9} \int_1^{28} \frac{du}{u} = \frac{1}{9} \ln u \Big|_1^{28} = \frac{1}{9} \ln 28.$$

5. Use L'Hôpital's rule to find the limit $\lim_{x \rightarrow 0} \frac{\cos x - \cos(3x)}{\sin^2 x}$.

Solution:

$$\lim_{x \rightarrow 0} \frac{\cos x - \cos 3x}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{-\sin x + 3 \sin 3x}{2 \sin x \cos x} = \lim_{x \rightarrow 0} \frac{-\cos x + 9 \cos 3x}{2 \cos^2 x - 2 \sin^2 x} = \frac{8}{2} = 4.$$

6. Use integration by parts to find the antiderivative $\int x^4 \ln x dx$.

Solution:

$$\int x^4 \ln x dx = \frac{1}{5} x^5 \ln x - \int \frac{1}{5} x^5 \frac{1}{x} dx = \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + C.$$

7. Use a trigonometric substitution to find the antiderivative $\int \frac{x^2}{(4-x^2)^{3/2}} dx$.

Solution: We substitute $x = 2 \sin t$ and obtain

$$\begin{aligned} \int \frac{x^2}{(4-x^2)^{3/2}} dx &= \int \frac{4 \sin^2 t \cdot 2 \cos t}{(2 \cos t)^3} dt = \int \tan^2 t dt \\ &= \tan t - t + C = \frac{x}{\sqrt{4-x^2}} - \arcsin \frac{x}{2} + C. \end{aligned}$$

8. Use partial fractions to find the antiderivative $\int \frac{x+1}{x^2-2x} dx$.

Solution: The partial fraction expansion is

$$\frac{x+1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}.$$

We find $A = -1/2$ and $B = 3/2$ from

$$x+1 = A(x-2) + Bx$$

by substituting $x = 0$ and $x = 2$. Therefore,

$$\frac{x+1}{x(x-2)} = \frac{-1/2}{x} + \frac{3/2}{x-2}.$$

The anti-derivative is

$$\int \frac{x+1}{x^2-2x} dx = -\frac{1}{2} \ln |x| + \frac{3}{2} \ln |x-2| + C.$$