Calculus and Analytic Geometry, Math 232
Review for Final Exam

12.1 Find the limit of the following sequences

$$a_n = (2n^3)^{1/n}, \quad b_n = \sin\left(\frac{n\pi}{n+1}\right), \quad c_n = \frac{3n^2+5n-3}{6n^2-7n+1}.$$

12.1 Determine whether the sequence $x_n = \ln(n+1) - \ln n$ is bounded and/or monotone.

12.2 Find the sum of the series $\displaystyle\sum_{k=3}^{\infty} \frac{1}{2^{k-1}}$.

12.2 Find the sum of the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-k}$.

12.3 Decide by the integral test whether the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^{3/2}}$ converges.

12.3 Estimate the sum $\displaystyle\sum_{k=10}^{\infty} \frac{1}{k^3}$ using integrals.

12.4 Decide by the comparison test whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{2k-1}{k^2+3}$ converges.

12.5 How many terms of the series $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k^3}$ do we need to find the sum with error less than 10^{-3}?

12.6 Decide if the series $\displaystyle\sum_{k=1}^{\infty} (-1)^k\frac{1}{k^2}$ converges conditionally, converges absolutely, or diverges.

Solution: We consider the series

$$\sum_{k=1}^{\infty} \left|(-1)^k\frac{1}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

This is a convergent p-series ($p=2$). Therefore, the series $\sum_{k=1}^{\infty}(-1)^k\frac{1}{k^2}$ converges absolutely.

12.8 Find the radius and the interval of convergence of the power series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k}x^k$.

Solution: We us the ratio test:

$$\frac{|a_{k+1}|}{|a_k|} = \frac{3^{k+1}k|x|^{k+1}}{3^k|x|^k(k+1)} = 3|x|\frac{k}{k+1} \to 3|x|.$$

The series converges absolutely if $3|x| < 1$ or $|x| < \frac{1}{3}$. The radius is $R = \frac{1}{3}$. At the right end point $x = a + R = 0 + \frac{1}{3} = \frac{1}{3}$ we get the series

$$\sum_{k=1}^{\infty} \frac{3^k}{k} \left(\frac{1}{3}\right)^k = \sum_{k=1}^{\infty} \frac{1}{k}.$$

This is the harmonic series which diverges. At the left end point $x = a - R = 0 - \frac{1}{3} = -\frac{1}{3}$ we get the series

$$\sum_{k=1}^{\infty} \frac{3^k}{k} \left(-\frac{1}{3}\right)^k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k}.$$

This is the alternating harmonic series which converges by the alternating series test. Therefore, the series $\sum_{k=1}^{\infty} \frac{3^k}{k} x^k$ converges for $-\frac{1}{3} \leq x < \frac{1}{3}$ and diverges for all other x.

12.9 Find the radius of the power series $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$. Apply term-by-term differentiation to find its sum.

Solution: To find the radius we use the ratio test:

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|x|^{k+1} k}{|x|^k (k+1)} = \frac{k}{k+1} |x| \to |x| < 1.$$

Therefore, the radius is $R = 1$. For $-1 < x < 1$ we define the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k.$$

By term-by-term differentiation,

$$f'(x) = \sum_{k=1}^{\infty} \frac{1}{k} k x^{k-1} = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}.$$

By anti-differentiation,

$$f(x) = -\ln(1-x) + C.$$

We find the constant C by evaluating at $x = 0$: $0 = f(0) = -\ln(1 - 0) + C = C$. Therefore, $C = 0$. The final result is

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k \quad \text{for } -1 < x < 1.$$

12.10 Find the third degree Taylor polynomial of the function $f(x) = \tan x$ at $x = 0$.

Solution: The nth degree Taylor polynomial is

$$P_n(x) = \sum_{k=0}^{n} c_k (x-a)^k, \quad c_k = \frac{f^{(k)}(a)}{k!}.$$

In this problem $f(x) = \tan x$, $a = 0$ and $n = 3$. We first compute the first 3 derivatives of $f(x)$:

$$f'(x) = 1 + \tan^2 x,$$
$$f''(x) = 2 \tan x(1 + \tan^2 x),$$
$$f'''(x) = 2(1 + \tan^2 x) + 6 \tan^2 x(1 + \tan^2 x)$$

Then

$$c_0 = f(0) = 0,\ c_1 = f'(0) = 1,\ c_2 = \frac{f''(0)}{2!} = 0,\ c_3 = \frac{f''(0)}{3!} = \frac{2}{6} = \frac{1}{3}.$$

The Taylor polynomial is

$$P_3(x) = 0 + 1 \cdot x + 0 \cdot x^2 + \frac{1}{3}x^3 = x + \frac{1}{3}x^3.$$