Calculus and Analytic Geometry Math 232

Homework 3

1. Find the limits of the sequences

$$a_n = \frac{\sqrt{7+3n+5n^2}}{3+2n}, \quad b_n = \frac{n^2}{2^n}.$$

Solution: By the limit laws, we get

$$a_n = \frac{\sqrt{7+3n+5n^2}}{3+2n} = \frac{\sqrt{\frac{7}{n^2}+\frac{3}{n}+5}}{\frac{3}{n}+2} \to \frac{\sqrt{0+0+5}}{0+2} = \frac{\sqrt{5}}{2}.$$

Using L'Hospital's rule, we get

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} \frac{x^2}{2^x} = \lim_{x\to\infty} \frac{2x}{2^x \ln 2} = \lim_{x\to\infty} \frac{2}{2^x (\ln 2)^2} = 0.$$

2. Evaluate the infinite sum

$$\sum_{k=1}^{\infty} \left(\frac{2^{k+1}+1}{3^k} \right).$$

Solution:

$$\sum_{k=1}^{\infty} \left(\frac{2^{k+1}+1}{3^k} \right) = \sum_{k=1}^{\infty} \frac{2^{k+1}}{3^k} + \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{4}{3}\frac{1}{1-\frac{2}{3}} + \frac{1}{3}\frac{1}{1-\frac{1}{3}} = 4 + \frac{1}{2} = \frac{9}{2}.$$

3. Evaluate the infinite sum (using "telescoping")

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)!}$$

Solution: We write

$$\frac{k}{(k+1)!} = \frac{(k+1)-1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}.$$

Then by telescoping

$$s_n = \sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1.$$

4. Give a lower and an upper estimate for the finite sum $\sum_{k=1}^{1000} \frac{1}{k^2}$ using the method of Section 12.3.

Solution: The upper bound is

$$\sum_{k=1}^{1000} \frac{1}{k^2} \le 1 + \int_1^{1000} \frac{dx}{x^2} = 1 + \left(-\frac{1}{1000} + 1\right) = 2 - \frac{1}{1000}.$$

and the lower bound is

$$\sum_{k=1}^{1000} \frac{1}{k^2} \ge \int_1^{1001} \frac{dx}{x^2} = 1 - \frac{1}{1001}.$$