DERIVING PROGNOSTIC EQUATIONS FOR CLOUD FRACTION AND LIQUID WATER CONTENT

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1. INTRODUCTION

A weather or climate model must prognose accurate moisture and temperature fields in part because these fields are used to drive parameterizations in the model. Driving the parameterizations is difficult because many microphysical and radiative processes are both nonlinear and highly variable. An example is autoconversion of cloud droplets to drizzle drops. For such processes, grid box *mean* quantities often lack sufficient information to accurately drive the parameterizations. Rather, the host model needs to provide information to the parameterization about subgrid-scale variability.

One approach to doing this is to formulate prognostic equations for the grid box mean of specific liquid water content, $\langle q_l \rangle$, and cloud fraction, C. This provides information about partial cloudiness. Tiedtke (1993) presented modeled equations for $\langle q_l \rangle$ and C, but he did not derive them from first principles. Wang and Wang (1999) did derive equations for C and $\langle q_l \rangle$, but they assumed a Gaussian family of FDFs. An FDF is the spatial average of a probability density function over a grid-box-sized volume (Colucci et al. 1998). The FDF, P_s, has all the properties of a PDF (Pope 2000, p. 59); that is, the FDF is non-negative everywhere and is normalized. Gregory et al. (2002), Wilson and Gregory (2003), and Bushell et al. (2003) derived equations for C and $\langle q_l \rangle$, but they were not completely general. For instance, the first two papers assume that forcings are applied uniformly across the grid box.

This paper derives prognostic equations for C and $\langle q_l \rangle$ that are valid for any FDF shape and for non-uniform forcing. The derivation starts from the evolution equation for the FDF of moisture. The FDF equation is a prognostic equation that can be derived from the conservation of moisture. This FDF equation and others like it have been widely studied by combustion engineers since the pioneering work of Lundgren (1967, 1969); see, for example, Pope (2000).

Deriving equations for C and $\langle q_l \rangle$ is useful because these equations contain exact but unclosed terms that can in principle be estimated by large eddy simulation and compared with parameterizations of the terms. Deriving C and $\langle q_l \rangle$ shows that closure of terms in the equations for C and $\langle q_l \rangle$ depends on and requires information about the relevant FDF. In addition, the C and $\langle q_l \rangle$ equations contain an unclosed dissipation term that does not appear in equations for conserved scalars. For this term, this paper compares a closure due to (Tiedtke 1993) versus another called the linear mean-square estimation (LMSE) model.

Interested readers may find more details about this research in Larson (2004).

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2. THE STARTING POINT: THE EVOLUTION EQUATION FOR THE FILTERED DENSITY FUNCTION

We choose to study the one-dimensional FDF of a variable s that was introduced by Mellor (1977) and Sommeria and Deardorff (1977). The variable s has several useful properties. First, when $s > 0$, $s \cong q_l$ (although s can be negative, whereas q_l cannot). Second, s is conserved during condensation (although not during adiabatic pressure changes). Third, s is a single variable that accounts for how liquid water varies with both total water content and temperature. The variable s has the same units as q_l, e.g. g kg^{-1}. The conservation equation for s is an advection-diffusion equation with a source term, S_s.

A prognostic transport equation for the FDF, P_s, can be rigorously derived from the conservation equation for s and other fundamental equations and definitions (Colucci et al. 1998; Klimenko and Bilger 1999; Pope 2000; Larson 2004). We assume the diffusivity of s, κ, is constant and that the fluid is Boussinesq. The final result is

$$\underbrace{\frac{\partial P_s}{\partial t}}_{\text{Time tendency}} + \underbrace{\frac{\partial}{\partial x_i}\left(\langle u_i|\psi\rangle P_s\right)}_{\text{Advection}} = \underbrace{\kappa\nabla^2 P_s}_{\text{Diffusion}} - \underbrace{\frac{\partial^2}{\partial\psi^2}\left(\left\langle \kappa \frac{\partial s}{\partial x_i}\frac{\partial s}{\partial x_i}\middle|\psi\right\rangle P_s\right)}_{\text{Dissipation}} - \underbrace{\frac{\partial}{\partial\psi}\left[\langle S_s|\psi\rangle P_s\right]}_{\text{Source}}. \quad (1)$$

We write an unconditional average or filtered value of a function $f(x_i,t)$ as $\langle f(x_i,t)\rangle$. The conditional average,

$\langle f|\psi\rangle$, is the average of $f(x_i,t)$ over those points for which $s(x_i,t)=\psi$.

In this equation, $P_s = P_s(\psi; x_i, t)$ depends on five independent variables: the usual ones of space (x_i) and time (t), plus the so-called "sample-space" variable, ψ, corresponding to s. The addition of the sample-space variable adds an extra dimension to the equation: the equation tracks how the filtered density $P_s(\psi; x_i, t)$ at each value of ψ evolves in space and time. In this sense, Eq. (1) differs from the conservation equation for $s(x_i,t)$ and resembles more the equation for bin microphysics, which bins hydrometeors according to radius and transports the bins individually in space and time.

We now comment on the terms in the FDF equation (1), starting from the left.

- The time tendency term quantifies how the FDF of s within the filtered region near x_i changes shape in sample space (ψ-space) with time.
- The advection term transports the filtered density P_s in physical space (x_i space), not by the full averaged velocity $\langle u_i\rangle$, but by $\langle u_i|\psi\rangle$, the velocity conditionally averaged with respect to the condition that $s=\psi$.
- The diffusion term diffuses P_s in physical space. Although the diffusion term cannot change the shape of P_s averaged over the whole domain, it can change the shape of P_s in a particular grid box.
- The dissipation term does change the shape of the domain-integrated FDF of s, unlike the aforementioned diffusion term. In fact, one can see that the dissipation term has the form of a Laplacian diffusion term in *sample space* (ψ-space) with a non-positive diffusivity, $-\langle\kappa(\partial s/\partial x_i)(\partial s/\partial x_i)|\psi\rangle$ (Chen and Kollmann 1994; Dopazo 1994). A negative diffusivity, rather than broadening the domain-integrated FDF, tends to narrow it in ψ-space by mixing fluid parcels with different values of s. Therefore the dissipation term, acting unopposed, homogenizes s.

3. DERIVING A PROGNOSTIC EQUATION FOR CLOUD FRACTION

We now derive a prognostic equation for the cloud fraction within a filtered region near x_i, e.g., within a grid box.

If all vapor in excess of saturation is converted immediately to liquid, then cloud fraction, C, is given by:

$$C(x_i,t) = \int_{-\infty}^{\infty} d\psi\ P_s(\psi; x_i, t) H(\psi) = \langle H(s)\rangle\,, \quad (2)$$

where H is the Heaviside step function. This formula states that the cloud fraction equals that portion of the area under P_s that corresponds to q_t in excess of saturation, i.e. the area with $s>0$.

The derivation proceeds simply by integrating the FDF equation over the cloudy portion of the domain. That is, we apply the operator

$$\int_{\psi=0}^{\psi=\infty} d\psi = \int_{\psi=-\infty}^{\psi=\infty} d\psi H(\psi) \quad (3)$$

to the FDF equation (1). We assume here that the fluid is Boussinesq and that the diffusivity κ is constant. Recalling that $C=\langle H(s)\rangle$, we find

$$\begin{aligned} \underbrace{\frac{\partial C}{\partial t}}_{\text{Time tendency}} &+ \underbrace{\frac{\partial}{\partial x_i}\langle u_i H(s)\rangle}_{\text{Advection}} \\ &= \underbrace{\kappa\nabla^2 C}_{\text{Diffusion}} \\ &\quad + \underbrace{\left[\frac{\partial}{\partial\psi}\left(P_s\left\langle \kappa\ \frac{\partial s}{\partial x_i}\frac{\partial s}{\partial x_i}\right|\psi\right\rangle\right)\right]_{\psi=0}}_{\text{Dissipation}} \\ &\quad + \underbrace{\left(\langle S_s|\psi\rangle\, P_s\right)_{\psi=0}}_{\text{Source}}. \end{aligned} \quad (4)$$

A similar equation for generic intermittent fields was obtained by Chen and Kollmann (1994). Several of the terms in the equation merit discussion.

- The advection term shows that cloud is advected by the within-cloud velocity, not the average full velocity $\langle u_i\rangle$ that is typically prognosed by a model.
- The dissipation term describes how mixing affects C by narrowing P_s. Mixing can either increase or decrease C. Consider the simple case in which $\langle\kappa(\partial s/\partial x_i)(\partial s/\partial x_i)|\psi\rangle$ is constant. Then the sign of the dissipation term depends on the slope of P_s at $\psi=0$, that is, at saturation. For instance, suppose P_s is a single Gaussian in ψ and that it narrows in time because of mixing (see Fig. 1). If $\langle s\rangle > 0$, so that C is large ($C > 1/2$), then $\partial P_s/\partial\psi|_{\psi=0} > 0$, implying that mixing tends to increase C. This is because mixing causes the unsaturated tail ($\psi<0$) to decrease in area. Likewise, if $\langle s\rangle < 0$, then C is small and mixing tends to decrease C.
- Computing the source term requires that we determine the average strength of the source at saturated points within the grid box, $(\langle S_s|\psi\rangle)_{\psi=0}$; knowledge of the grid box average source, $\langle S_s\rangle$, is not sufficient.

Both the dissipation and source terms depend explicitly on the FDF, P_s; the advection term depends implicitly on P_s. Therefore the equation for C cannot be closed without an explicit or implicit model for P_s. Hence if one chooses to predict cloud fraction by prognosing C, one does not thereby circumvent the difficulties involved in predicting P_s.

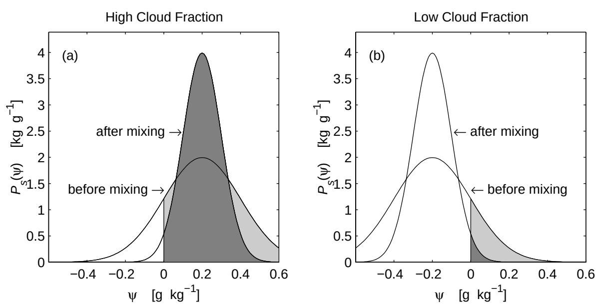

Figure 1: Gaussian FDFs of s, $P_s(\psi)$, before and after a hypothetical dissipative mixing process occurs for (a) high cloud fraction, and (b) low cloud fraction. The light grey shading indicates the cloud fraction before mixing, and the dark grey, the cloud fraction after mixing. For high cloud fraction, the mean of P_s is at $\psi > 0$. Then mixing, which narrows the FDF, *increases* cloud fraction. For low cloud fraction, however, the opposite is true.

4. THE DISSIPATION TERM IN THE CLOUD FRACTION EQUATION: A DIFFICULT PARAMETERIZATION PROBLEM

The dissipation term alters C by decreasing the variance of s. Teixeira (2001) found that in subtropical boundary layers, the dissipation term is one of the largest terms in Tiedtke's (1993) modeled equation for C.

One simple closure, widely used in the turbulence and combustion literature, is the linear mean-square estimation (LMSE) model (Villermaux and Devillon 1972; Dopazo and O'Brien 1974; Pope 2000, Eq. 12.326). The LMSE closure has the defect of preserving the shape of the FDF as it narrows, rather than relaxing it to a Gaussian (Pope 2000, p. 550). If the LMSE diffusion model acts on a skewed FDF, the FDF does not become symmetrical. The LMSE closure for both the diffusion and dissipation terms in the equation for C is

$$\underbrace{\kappa\nabla^2 C}_{\text{Diffusion}} + \underbrace{\left[\frac{\partial}{\partial\psi}\left(P_s\left\langle \kappa\,\frac{\partial s}{\partial x_i}\frac{\partial s}{\partial x_i}\right|\psi\right\rangle\right)\right]_{\psi=0}}_{\text{Dissipation}} = \left(P_s\frac{1}{2}c_s\frac{\epsilon}{e}\langle s\rangle\right)_{\psi=0}. \quad (5)$$

This parameterization leaves C unaltered if $P_s|_{\psi=0} = 0$, that is, if there are no parcels just at saturation. Otherwise, C increases if $\langle s\rangle > 0$, and C decreases if $\langle s\rangle < 0$. This conforms to the expected effect of mixing on Gaussian FDFs shown in Fig. 1. The expression $(1/2)c_s\epsilon/e$ may be thought of as an inverse time scale that depends on the turbulent dissipation and influences the rapidity with which C dissipates.

Tiedtke (1993, Eq. 32) has proposed another expression for the dissipation of cloud fraction:

$$\text{Dissipation} \cong \frac{C^2}{\langle q_l\rangle}K(\langle q_s\rangle - \langle q\rangle) \geq 0, \quad (6)$$

where q is the specific humidity, q_s is the saturation specific humidity, and $K = 10^{-6}$ s^{-1} is an inverse time constant. We can pass from the LMSE model of dissipation of C, (5), to Tiedtke's model, (6), if we change the following. We assume that P_s is a uniform (rectangular) FDF. We equate timescales, so that $K = (1/4)c_s\epsilon/e$. Finally we replace $\langle s\rangle$ by $(q_s - q)$. The last change, however, is significant, since the LMSE model allows dissipation to either increase or decrease C, depending on the cloud fraction, the relative humidity of unsaturated air, and the liquid water content of cloudy air (see Fig. 1). Tiedtke's (1993) formula, in contrast, only permits dissipation to decrease C. Although this is a drawback, we note that in the cases in which dissipation increases cloud fraction, q approaches q_s, so that Tiedtke's expression (6) diminishes C weakly, not strongly.

5. DERIVING A PROGNOSTIC EQUATION FOR LIQUID WATER

We now use the FDF equation (1) to derive a prognostic equation for the filtered specific liquid water content, $\langle q_l\rangle$. We again assume that any supersaturation is immediately converted to liquid. Then q_l is given, in terms of s, as $q_l = sH(s)$, and $\langle q_l\rangle$ is

$$\langle q_l\rangle = \int_{-\infty}^{\infty} d\psi\; P_s(\psi; x_i, t)\psi H(\psi) = \langle sH(s)\rangle\,. \quad (7)$$

We can derive an equation for $\langle q_l\rangle$ by applying the operator

$$\int_{\psi=0}^{\psi=\infty} d\psi\;\psi = \int_{-\infty}^{\infty} d\psi\; H(\psi)\psi \quad (8)$$

to Eq. (1) for P_s. Assuming a Boussinesq flow, and making the questionable assumption that the diffusivity κ is constant, we obtain

$$\underbrace{\frac{\partial\langle q_l\rangle}{\partial t}}_{\text{Time tendency}} + \underbrace{\frac{\partial}{\partial x_i}\langle u_i q_l\rangle}_{\text{Advection}} = \underbrace{\kappa\nabla^2\langle q_l\rangle}_{\text{Diffusion}} - \underbrace{\left(P_s\left\langle \kappa\,\frac{\partial s}{\partial x_i}\frac{\partial s}{\partial x_i}\right|\psi\right\rangle\right)\Bigg|_{\psi=0}}_{\text{Dissipation}} + \underbrace{\langle S_s H(s)\rangle}_{\text{Source}}. \quad (9)$$

The terms are somewhat analogous to their counterparts in the prognostic equation for C (4). We only note that the dissipation term describes how diffusive mixing, which tends to homogenize the fluid, affects

$\langle q_l \rangle$ via evaporation or condensation. Since both P_s and the scalar dissipation, $\kappa(\partial s/\partial x_i)(\partial s/\partial x_i)$, are nonnegative, the dissipation term cannot increase $\langle q_l \rangle$. This is a consequence of the fact that we have assumed that s mixes linearly. In contrast, dissipation or mixing can either increase or decrease C readily (recall Eq. 4); therefore, mixing has different effects on $\langle q_l \rangle$ and C. This is an important consideration for modeling purposes.

6. CONCLUSIONS

The main result of this paper is the derivation of the prognostic equation (4) for cloud fraction, C, and equation (9) for filtered specific liquid water content, $\langle q_l \rangle$. These equations were derived from the prognostic equation for the filtered density function (FDF) of s, (1). An extra term emerges in the C and $\langle q_l \rangle$ equations, a term that does not appear in equations for variables that are conserved with respect to condensation, such as total specific water content, q_t. The extra term arises from the tendency of dissipation or diffusive mixing to homogenize a fluid. From inspection of the dissipation term, we see that it always diminishes $\langle q_l \rangle$ with the diffusive model we use. In contrast, dissipation can either increase or decrease C.

We have compared a closure for dissipation used by combustion engineers with the closure used by Tiedtke (1993). Perhaps the biggest difference is that the engineers' linear mean-square estimation (LMSE) model can either increase or decrease C, but Tiedtke's parameterization can only decrease C. In nature, either an increase or decrease in C is possible (see Fig. 1). The dissipation term in Tiedtke's equation for C is a dominant term in many situations (Teixeira 2001), although the term tends to zero as the grid box average specific humidity approaches saturation.

Should modelers prognose moments of s and diagnose C and q_l based on an assumed FDF, or should they prognose C and q_l directly? Each method has disadvantages. However, many of the terms in the C and $\langle q_l \rangle$ equations depend explicitly or implicitly on the FDF; therefore, prognosing C and $\langle q_l \rangle$ does not evade the problem of prognosing or diagnosing the FDF.

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