Characterizing Option Prices by Linear Programs

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Abstract. The price of various options on a risky asset are characterized via a linear program involving the generator(s) of the processes involved and corresponding occupation measure(s). Examples include the price of European, American, barrier, lookback and perpetual Russian options.

1. Introduction

The options market exploded following the publication of the Black-Scholes pricing formula for a European option [**BS**]. This closed form expression provided practitioners with a simple method to obtain an approximate price for such options. Naturally, the option pricing formula depended on some simplifying assumptions, such as the asset prices following a geometric Brownian motion with constant coefficients and the market being frictionless. The resulting model, therefore, did not capture some important aspects of the real world.

Improvements to the model are more complex and less tractable. For example, the inclusion of proportional transaction costs in portfolio management in the maximization of expected utility by Davis and Norman [**DN**] resulted in transactions occurring according to the local time of a diffusion process on buy and sell boundaries of a no transaction region, with these boundaries being determined from the numerical solution of a differential equation (see also Shreve and Soner [**ShrSon**]). Models involving fixed transaction costs involve impulse control, and variational and quasi-variational inequalities (see for example, [**EH**], [**AMS**]). Cadenillas [**C**] surveys the results concerning transaction cost problems.

At the same time, different types of options have been demanded by customers to meet their investment requirements. Simple examples of such options include American, Asian, barrier and lookback options. The pricing of such options poses a challenge to theorists and practitioners alike.

This paper illustrates a different approach to the formulation of option prices through the characterization of such prices in terms of the solution of a linear program. The linear program takes as its variables the occupation measures corresponding to the evolution of the asset prices and auxiliary processes as needed.

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The paper begins with a formulation of the model in terms of general stochastic processes. It then presents the linear programming formulation for the European and American options under the geometric Brownian motion model to display the basic linear programming approach. Next, the case of barrier options is discussed, followed by consideration of lookback options. The paper concludes with the formulation for a perpetual Russian option.

2. General Model

In this section, we use a very general formulation for the processes under consideration. This formulation is flexible enough to cover most models of interest in option pricing. We also display a fundamental existence result which provides the foundation for the equivalence of the characterization of the option price in terms of a linear program and the well-known characterization as the expectation of the option under an equivalent martingale measure.

We establish the formulation and stated the existence result in full generality, in particular, with controls included. For the purposes of this paper, the control space is not needed so one can fix a particular singleton control space $U = \{u\}$ and thereby reduce the formulation and result. We have chosen to exhibit the more general statements in order to be able to refer to them in the concluding remarks.

2.1. Stochastic Process Formulation.

For a complete, separable, metric space S, we define $M(S)$ to be the space of Borel measurable functions on S, $B(S)$ to be the space of bounded, measurable functions on S, $C(S)$ to be the space of continuous functions on S, $\overline{C}(S)$ to be the space of bounded, continuous functions on S, $\mathcal{M}(S)$ to be the space of finite Borel measures on S, and $\mathcal{P}(S)$ to be the space of probability measures on S. $\mathcal{M}(S)$ and $\mathcal{P}(S)$ are topologized by weak convergence.

Let $\mathcal{L}_t(S) = \mathcal{M}(S \times [0,t])$. We define $\mathcal{L}(S)$ to be the space of measures ξ on $S \times [0,\infty)$ such that $\xi(S \times [0,t]) < \infty$, for each t, and topologized so that $\xi_n \to \xi$ if and only if $\int f d\xi_n \to \int f d\xi$, for every $f \in \overline{C}(S \times [0,\infty))$ with $\text{supp}(f) \subset S \times [0,t_f]$ for some $t_f < \infty$. Let $\xi_t \in \mathcal{L}_t(S)$ denote the restriction of ξ to $S \times [0,t]$.

Let E denote the state space of the process and U denote the space of controls. For the purposes of this paper, E and U are typically $\mathbb{R}^d$ for some value of d.

Let $A, B : \mathcal{D} \subset \bar{C}(E) \to C(E \times U)$ and $\nu_0 \in \mathcal{P}(E)$. Let (X, Λ) be an $E \times \mathcal{P}(U)$-valued process and Γ be an $\mathcal{L}(E \times U)$-valued random variable. Let Γ_t denote the restriction of Γ to $E \times U \times [0,t]$. Then (X, Λ, Γ) is a relaxed solution of the *singular, controlled martingale problem* for (A, B, ν_0) if there exists a filtration $\{\mathcal{F}_t\}$ such that (X, Λ, Γ_t) is $\{\mathcal{F}_t\}$-progressive, $X(0)$ has distribution ν_0, and for every $f \in \mathcal{D}$,

$$f(X(t)) - \int_0^t \int_U Af(X(s),u)\Lambda_s(du)ds - \int_{E\times U\times[0,t]} Bf(x,u)\,\Gamma(dx \times du \times ds) \tag{2.1}$$

is an $\{\mathcal{F}_t\}$-martingale. In this expression, X is the state process, Λ is a random selection of values which act as "controls" on X, A is the "absolutely continuous" generator, B is the "singular" generator and Γ is a random measure which summarizes the singular behavior of the state and/or control. We assume that A and B have the following properties.

Condition 2.1.

i) $A, B : \mathcal{D} \subset \overline{C}(E) \to C(E \times U)$, $1 \in \mathcal{D}$, *and* $A1 = 0, B1 = 0$.

ii) *There exist* $\psi_A, \psi_B \in C(E \times U)$, $\psi_A, \psi_B \geq 1$, *and constants* a_f, b_f, $f \in \mathcal{D}$, *such that*

$$|Af(x,u)| \leq a_f \psi_A(x,u), \qquad |Bf(x,u)| \leq b_f \psi_B(x,u), \qquad \forall (x,u) \in \mathcal{U}.$$

iii) *Defining* $(A_0, B_0) = \{(f, \psi_A^{-1} Af, \psi_B^{-1} Bf) : f \in \mathcal{D}\}$, (A_0, B_0) *is* separable *in the sense that there exists a countable collection* $\{g_k\} \subset \mathcal{D}$ *such that* (A_0, B_0) *is contained in the bounded, pointwise closure of the linear span of* $\{(g_k, A_0 g_k, B_0 g_k) = (g_k, \psi_A^{-1} A g_k, \psi_B^{-1} B g_k)\}$.

iv) *For each* $u \in U$, *the operators* A_u *and* B_u *defined by* $A_u f(x) = Af(x,u)$ *and* $B_u f(x) = Bf(x,u)$ *are pre-generators.*

v) $\mathcal{D}$ *is closed under multiplication and separates points.*

EXAMPLE 2.2. We illustrate a financial model having all the aspects of the general model in (2.1), though not all of these parts will be needed in the pricing of options.

Consider a portfolio model consisting of processes X_1, representing the amount invested in a riskless bond, and X_2, giving the amount invested in a risky asset. At time t, assume the bond earns interest at rate $r(t)$ and the asset return has mean appreciation rate $\mu(t)$ and volatility $\sigma(t)$. Let $c(t)$ denote the rate of consumption at time t and assume that purchases and sales of the risky asset incur fixed costs. The processes X_1 and X_2 therefore satisfy

$$\begin{aligned} dX_1(t) &= [r(t)X_1(t) - c(t)]\,dt - [u(t-) + \delta]d\xi(t) \\ dX_2(t) &= X_2(t)\,[\mu(t)\,dt + \sigma(t)\,dW(t)] + u(t-)d\xi(t), \end{aligned}$$

where W is a Brownian motion, ξ is a counting process which records the occurrances of trades, $u(t)$ denotes the amount of the trade at time t and δ is the fixed cost for the transactions (which is paid out of the bond). We assume $(X_1(0), X_2(0)) = (x_0, y_0)$.

It then follows that the absolutely continuous generator A tracks the behavior of the portfolio between transactions

$$Af(x_1,x_2,c) = (r(t)x_1 - c)f_{x_1}(x_1,x_2) + \mu(t)x_2 f_{x_2}(x_1,x_2) + (1/2)\sigma^2(t)x_2^2 f_{x_2x_2}(x_1,x_2)$$

whereas the singular generator B accounts for the transactions

$$Bf(x_1,x_2,u) = f(x_1 - u - \delta, x_2 + u) - f(x_1, y_2).$$

The "controls" available to the portfolio manager are the rate of consumption c, and the times and amounts of purchases and sales of the asset.

2.2. Occupation measures and the basic adjoint relation. Let T denote a (finite) time horizon and observe that (2.1) being a martingale implies that for each $f \in \mathcal{D}$

$$\begin{aligned} E\Bigg[f(X(T)) &- \int_0^T \int_U Af(X(s),u)\Lambda_s(du)\,ds \\ &- \int_{[0,T]\times E \times U} Bf(x,u)\Gamma(ds \times dx \times du) \Bigg] = E[f(X(0))]. \end{aligned} \tag{2.2}$$

Recall ν_0 denotes the initial distribution of X. Let ν_T denote the distribution of $X(T)$, μ_0 denote the expected "absolutely continuous" occupation measure defined by

$$\mu_0(G) = E\left[\int_0^T \int_U I_G(X(s),u)\Lambda_s(du)\,ds\right], \qquad G \in \mathcal{B}(E\times U), \tag{2.3}$$

and μ_1 be the expected "singular" occupation measure

$$\mu_1(G) = E\left[\int_{[0,T]\times E\times U} I_G(x,u)\Gamma(ds\times dx\times du)\right], \qquad G \in \mathcal{B}(E\times U). \tag{2.4}$$

Observe that the total mass of μ_0 is T. It then follows from (2.2) that for every $f \in \mathcal{D}$

$$\int f\,d\nu_T - \int Af\,d\mu_0 - \int Bf\,d\mu_1 = \int f\,d\nu_0. \tag{2.5}$$

This relation will be used in the pricing of European options, which can only be exercised at the final time T. We refer to identity (2.5) and appropriate variations of it as the basic adjoint relation.

We obtain the corresponding adjoint relation needed for pricing American options, which can be exercised any time up to the expiration time T, as follows. Let τ denote the exercise time of the option and note that $\tau \le T$. The optional sampling theorem implies that T can be replaced by τ in (2.2). Letting ν_τ denote the distribution of $X(\tau)$ and similarly modifying the definition of the occupation measures μ_0 (see (3.4) below) and μ_1 then yields for every $f \in \mathcal{D}$

$$\int f\,d\nu_\tau - \int Af\,d\mu_0 - \int Bf\,d\mu_1 = \int f\,d\nu_0. \tag{2.6}$$

Note that $\mu_0(E\times U) \le T$.

2.3. Existence result.

Starting with processes X, Λ and random measure Γ, it follows that the basic adjoint relations (2.5) and (2.6) hold. In Kurtz and Stockbridge [**KS2**], it is shown that the converse holds. We state the result for (2.6) but refer the reader to [**KS2**] for additional (mild) technical conditions.

THEOREM 2.3. *Let $\nu_0, \nu_\tau \in \mathcal{P}(E)$, μ_0 and μ_1 be finite measures on $E\times U$ satisfying (2.6). For $i = 0, 1$, let the transition functions η_i and marginal distributions μ_i^E satisfy $\mu_i(dx\times du) = \eta_i(x,du)\mu^E(dx)$. Then there exist a process X adapted to a filtration $\{\mathcal{F}_t\}$, a random measure Γ on $E\times[0,\infty)$ and an $\{\mathcal{F}_t\}$-stopping time τ such that*

$$\begin{aligned} f(X(t\wedge\tau)) \;&-\; \int_0^{t\wedge\tau}\int_U Af(X(s),u)\eta_0(X(s),du)ds \\ &-\; \int_{E_0\times[0,t\wedge\tau)}\int_U Bf(x,u)\eta_1(x,du)\Gamma(dx\times ds) \end{aligned} \tag{2.7}$$

is an $\{\mathcal{F}_t\}$-martingale for every $f \in \mathcal{D}$, and

$$\begin{aligned} &E\Bigg[\int_0^\tau \int_U c_0(X(s),u)\eta_0(X(s),du)ds \\ &\quad + \int_{E\times[0,\tau]}\int_U c_1(x,u)\eta_1(x,du)\Gamma(dx\times ds) + c_2(X(\tau))\Bigg] \\ &= \int_{E\times U} c_0(x,u)\,\mu_0(dx\times du) + \int_{E\times U} c_1(x,u)\,\mu_1(dx\times du) + \int_E c_2(x)\mu_2(dx) \end{aligned} \tag{2.8}$$

for every $c_0, c_1 \in M(E \times U)$ *and* $c_2 \in M(E)$ *that are bounded below.*

3. European and American Options

To begin with the pricing of options, consider a complete market consisting of a single bond and a single stock in which the prices satisfy

$$dB(t) = r(t)B(t)\,dt \tag{3.1}$$

$$dS(t) = S(t)[\mu(t)\,dt + \sigma(t)\,dW(t)]. \tag{3.2}$$

Let $T < \infty$ denote the expiration time of the option and let $C(S(T))$ denote the value of a contingent claim at the time of expiration. We assume throughout the paper that C is measurable and bounded below.

3.1. European option.

It is well-known that the arbitrage free price of C is given by $E^Q[C(S(T))]$, where Q is the unique equivalent martingale measure. Using linear programming, we characterize Q in terms of its expected occupation measure and use this measure and the corresponding distribution ν_T of $S(T)$ to specify this price.

First observe that S is a martingale under Q, hence S satisfies

$$dS(t) = \sigma(t)S(t)\,d\tilde{W}(t) \tag{3.3}$$

for some Q-Brownian motion process $\tilde{W}$. The generator of S under Q is

$$Af(s) = (1/2)\sigma^2(t)s^2 f''(s),$$

defined for all $f \in \mathcal{D} = C^2(\mathbb{R}^+)$. (Note since the stochastic process is the price of the stock, we use s rather than x to denote the state.) The sole generator for this model is the absolutely continuous generator A acting on functions of the state alone. It follows that for the process S of (3.3) the corresponding measures ν_0, ν_T, and μ_0 satisfy the suitably simplified version of (2.5). Conversely, for any ν_0 and for any pair of measures (ν_T, μ_0) satisfying (2.5), there exists a process S such that $f(S(t)) - \int_0^t Af(S(v))\,dv$ is a martingale. Under non-degeneracy conditions on $\sigma(t)$, this implies S satisfies (3.3).

Recalling that the option price is given by $E^Q[C(S(T))] = \int C(s)\nu_T(ds) =: \langle C, \nu_T\rangle$, we have the following characterization of the option price.

Theorem 3.1. *The price of the European option is the solution of the linear program*

$$\begin{array}{rl} \textit{Optimize} & \langle C, \nu_T\rangle \\ \textit{Subj. to} & \langle f, \nu_T\rangle - \langle Af, \mu_0\rangle = \langle f, \nu_0\rangle, \quad \forall f \in \mathcal{D}, \\ & |\nu_T| = 1, |\mu_0| = T, \end{array}$$

where $|\nu_T|$ *and* $|\mu_0|$ *denote the total masses of the measures* ν_T *and* μ_0*, respectively.*

Since the measure Q is unique, the feasible set of measures (ν_T, μ_0) consists of a singleton and it does not matter whether a maximization or minimization is performed.

3.2. American option. Only a small change is needed in the linear program of Theorem 3.1 in order to characterize an American option for C having expiration time T. Let τ denote an exercise time for the option, let ν_τ denote the corresponding distribution of $S(\tau)$ and μ_0 denote the expected occupation measure up to time τ:

$$\mu_0(G) = E\left[\int_0^\tau I_G(S(v))\,dv\right], \qquad G \in \mathcal{B}(\mathbb{R}^+). \tag{3.4}$$

It follows that (2.6) is satisfied by ν_0, ν_τ and μ_0. Moreover the same argument as in the previous subsection implies the existence of a solution to (3.3) up to a stopping time τ for each ν_0 and pair (ν_τ, μ_0) satisfying (2.6). The option price is then characterized by the following linear program.

Theorem 3.2. *The price of the American option is the solution of the linear program*

$$\begin{array}{rl} \textit{Maximize} & \langle C, \nu_\tau\rangle \\ \textit{Subj. to} & \langle f, \nu_\tau\rangle - \langle Af, \mu_0\rangle = \langle f, \nu_0\rangle, \quad \forall f \in \mathcal{D}, \\ & |\nu_\tau| = 1, |\mu_0| \le T. \end{array}$$

Remark 3.3. A standard approach to pricing American options is to reformulate the problem in terms of a free boundary problem. The free boundary remains an integral part of the problem in the linear program in that the measure ν_τ is concentrated on the stopping locations. Thus the linear program optimizes over all potential boundaries.

Remark 3.4. It is interesting to note that the only differences between Theorem 3.1 and Theorem 3.2 lie in the conditions $|\mu_0| = T$ and $\mu_0 \le T$ and the requirement that the linear program for the American option be a maximization problem.

4. Barrier Options

The linear programming characterization can be applied to price knock-out and knock-in options. Again, suppose that the prices follow the model (3.1), (3.3) under some martingale measure Q and a barrier option C with maturity T needs to be priced. Throughout this section, the barriers are s_1 and s_2, with $0 \le s_1 < s_2 \le \infty$. The cases of an "up and out" option and a "down and out" option are treated simultaneously, followed by the simultaneous treatment of the knock-in options.

4.1. European knock-out options. Assume that the option ceases, possibly returning a rebate R, when the asset price $S(t)$ leaves the interval (s_1, s_2). In this case, the option C takes the form

$$C(s) = C(s)I_{(s_1,s_2)}(s) + RI_{(s_1,s_2)^c}(s).$$

Consider the generator for a new process $\tilde{S}$ given by

$$\tilde{A}f(s) = (1/2)\sigma^2(t)s^2 f''(s)I_{(s_1,s_2)}(s). \tag{4.1}$$

Assuming that the distribution of the initial stock price is concentrated on (s_1, s_2), it follows that $\tilde{S}$ agrees with the actual stock price S until the random time τ at which both S and $\tilde{S}$ first exit the interval (s_1, s_2). After this the generator $\tilde{A}$ of the price process $\tilde{S}$ becomes 0 indicating that the process $\tilde{S}$ stops at the value s_1 or s_2. Observe that the European option $C(\tilde{S}(T))$ gives the value of the desired option.

THEOREM 4.1. *The price of the knock-out barrier option C having barriers $\{s_1, s_2\}$ is the solution of the linear program*

$$\begin{array}{rl} \textit{Optimize} & \langle C, \nu_T\rangle \\ \textit{Subj. to} & \langle f, \nu_T\rangle - \langle \tilde{A}f, \mu_0\rangle = \langle f, \nu_0\rangle, \quad \forall f \in \mathcal{D}, \\ & |\nu_T| = 1, |\mu_0| = T. \end{array}$$

Notice that the measures ν_T and μ_0 are concentrated on the closed interval $[s_1, s_2]$ and that both measures will have point masses at $\{s_1\}$ and $\{s_2\}$. The mass of ν_T at the points gives the probability that the process S exits (s_1, s_2) by the expiration time T.

4.2. American knock-out options. The characterization for an American knock-out barrier option modifies the corresponding European option in the same manner as an American option adjusts a European option. Let $\tau_1 = \inf\{t \geq 0 : S(t) \neq (s_1, s_2)\}$ denote the stopping time when a barrier is crossed so the option ceases. Let τ_2 denote the stopping time when the holder of the option chooses to exercise the option. Then setting $\tau = \tau_1 \wedge \tau_2$ in (2.6) and applying Theorem 2.3 yields the following characterization for the price of the option.

THEOREM 4.2. *The price of the American knock-out option is the solution of the linear program*

$$\begin{array}{rl} \textit{Maximize} & \langle C, \nu_\tau\rangle \\ \textit{Subj. to} & \langle f, \nu_\tau\rangle - \langle \tilde{A}f, \mu_0\rangle = \langle f, \nu_0\rangle, \quad \forall f \in \mathcal{D}, \\ & |\nu_\tau| = 1, |\mu_0| \leq T. \end{array}$$

4.3. Knock-in options. The approach to modeling the stock price process for the knock-out options was to modify the process so that it sticks at the barrier and evaluate the option using this new process. To handle the knock-in options requires a different approach using a singular operator.

Assume the market is modelled by (3.1), (3.3). As before, define the stopping time $\tau = \inf\{t \geq 0 : S(t) \neq (s_1, s_2)\}$ to be the time that the stock price hits the barrier. Consider the process ξ given by

$$\xi(t) = I_{[\tau,\infty)}(t),$$

noting that ξ is 0 until the barrier is reached, then jumps to 1 and remains there. Let $\mathcal{D}$ consist of functions $f : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$ which are twice continuously differentiable in the first variable. Define the operators $Af(s, \xi) = (1/2)\sigma^2(t)s^2 f_{ss}(s, \xi)$ and $Bf(x, \xi) = f(s, \xi + 1) - f(x, \xi)$. It follows that

$$f(S(t), \xi(t)) - \int_0^t Af(S(v), \xi(v))\, dv - \int_0^t Bf(S(v), \xi(v))\, d\xi(v)$$

is a martingale.

We now consider the option. Suppose that the option is C if the stock price process hits the barrier and returns a rebate of R if it fails to reach the barrier. Since the value of ξ identifies whether or not the price reaches the barrier, we can define the option as

$$\tilde{C}(s, \xi) = \begin{cases} R & \text{if } \xi = 0, \\ C(s) & \text{if } \xi = 1. \end{cases}$$

Applying the same reasoning yields the following characterization of knock-in options.

Theorem 4.3. *The price of a European knock-in barrier option C having barriers $\{s_1, s_2\}$ is the solution of the linear program*

$$\begin{array}{rl} \textit{Optimize} & \langle \tilde{C}, \nu_T \rangle \\ \textit{Subj. to} & \langle f, \nu_T \rangle - \langle Af, \mu_0 \rangle - \langle Bf, \mu_1 \rangle = \langle f, \nu_0 \rangle, \quad \forall f \in \mathcal{D}, \\ & |\nu_T| = 1, |\mu_0| = T, |\mu_1| < \infty. \end{array}$$

The modification of the argument for American options gives the following characterization.

Theorem 4.4. *The price of an American knock-in barrier option C having barriers $\{s_1, s_2\}$ is the solution of the linear program*

$$\begin{array}{rl} \textit{Maximize} & \langle \tilde{C}, \nu_\tau \rangle \\ \textit{Subj. to} & \langle f, \nu_\tau \rangle - \langle Af, \mu_0 \rangle - \langle Bf, \mu_1 \rangle = \langle f, \nu_0 \rangle, \quad \forall f \in \mathcal{D}, \\ & |\nu_\tau| = 1, |\mu_0| \le T, |\mu_1| < \infty. \end{array}$$

5. Lookback Options

The goal of this section is to identify a linear program for the price of a lookback option; that is, an option on the maximum price obtained by the asset over the life of the option.

In addition to the stock price process S satisfying (3.3), it is necessary to keep track of the maximum price to date. We denote this running maximum price process by

$$Y(t) = \max_{0 \le v \le t} S(t). \tag{5.1}$$

The pair (S, Y) is a Markov process in which the process Y is constant during excursions of the process S below its current maximum and increases only at times when S hits the maximum again. Thus the pair process only occupies the left-half-space above the diagonal $s = y$. The fact that Y increases only at times when $S(t) = Y(t)$ (a set that is singular with respect to Lebesgue measure) indicates that the reformulation will involve the singular generator. The running maximum process has been studied in [**HeiS**].

The common domain for the generators is $\mathcal{D} = \{f \in C^{2,0}(\mathbb{R}^2) : f_y(s, s) \text{ exists}\}$. The absolutely continuous generator is

$$Af(s, y) = (1/2)\sigma^2(t) s^2 f_{ss}(s, y)$$

which captures the behavior of the stock price during excursions. The singular generator is

$$Bf(s, y) = f_y(s, y)$$

which is active only when the two-dimensional process is on the diagonal $s = y$. Thus for each $f \in \mathcal{D}$,

$$f(S(t), Y(t)) - \int_0^t Af(S(v), Y(v))\, dv - \int_0^t Bf(S(v), Y(v))\, d\xi(v) \tag{5.2}$$

is a martingale, where ξ is a local time of the process S on the diagonal.

Defining the expected occupation measures as in (2.3) and (2.4) and letting ν_T denote the joint distribution of $(S(T), Y(T))$, it follows that (2.5) holds. The application of Theorem 2.3 with $\tau = T$ then implies that for any initial distribution ν_0 and measures ν_T, μ_0 and μ_1 satisfying (2.5), there exist processes S, Y and random measure Γ such that (2.7) is a martingale and for which (2.8) holds. The following result therefore holds.

THEOREM 5.1. *The price of the lookback option C is the solution of the linear program*

$$\begin{array}{ll} \textit{Optimize} & \langle C, \nu_T \rangle \\ \textit{Subj. to} & \langle f, \nu_T \rangle - \langle Af, \mu_0 \rangle - \langle Bf, \mu_1 \rangle = \langle f, \nu_0 \rangle, \quad \forall f \in \mathcal{D}, \\ & |\nu_T| = 1, |\mu_0| = T, |\mu_1| < \infty. \end{array}$$

6. Perpetual Russian Options

A perpetual Russian option is similar to a lookback option in that it depends on the historical prices of the asset. However, it has no expiration time; rather, it allows the holder to exercise the option at any time in the future. The payoff is a function of the current price and historical prices, discounted at a rate α. We consider the case where the option is a function of the maximum price to date and the current price.

Assume the market is modelled by (3.1), (3.3) with the maximum asset price process being given by (5.1) so that (5.2) is a martingale for each $f \in \mathcal{D}$. We require f to be bounded as well. It then follows that

$$\begin{aligned} e^{-\alpha t} f(S(t), Y(t)) \quad &- \quad \int_0^t e^{-\alpha v} [Af(S(v), Y(v)) - \alpha f(S(v), Y(v))] \, dv \\ &- \quad \int_0^t e^{-\alpha v} Bf(S(v), Y(v)) \, d\xi(v) \end{aligned}$$

is also a martingale for every $f \in \mathcal{D}$. Letting τ denote the exercise time, the value of the option at time τ is $e^{-\alpha\tau} C(S(\tau), Y(\tau))$; for example one such option is $e^{-\alpha\tau}[Y(\tau) - S(\tau)]$.

Let $\widehat{A}f(s,y) = Af(s,y) - \alpha f(s,y)$, define the *discounted* occupation measures μ_0 and μ_1 up to time τ by

$$\begin{aligned} \mu_0(G) &= E\left[\int_0^\tau e^{-\alpha v} I_G(S(v), Y(v)) \, dv\right], \qquad G \in \mathcal{B}(\mathbb{R}^2) \\ \mu_1(G) &= E\left[\int_0^\tau e^{-\alpha v} I_G(S(v), Y(v)) \, d\xi(v)\right], \quad G \in \mathcal{B}(\mathbb{R}^2) \end{aligned}$$

and define the *discounted* distribution of $(S(\tau), Y(\tau))$ by

$$\nu_\tau(G) = E\left[e^{-\alpha\tau} I_G(S(\tau), Y(\tau))\right], \qquad G \in \mathcal{B}(\mathbb{R}^2).$$

Note that the total mass of ν_τ is less than 1, the mass of μ_0 is bounded above by α^{-1} and μ_1 is concentrated on the diagonal $y = s$. It then follows as in Section 2.2 that

$$\int f \, d\nu_\tau - \int \widehat{A}f \, d\mu_0 - \int Bf \, d\mu_1 = \int f d\nu_0, \qquad \forall f \in \mathcal{D}.$$

Also observe that $E[e^{-\alpha\tau} C(S(\tau), Y(\tau))] = \int C \, d\nu_\tau$. A discounted version of Theorem 2.3 (also in [**KS2**]) then implies the following characterization for a perpetual Russian option.

THEOREM 6.1. *The price of a perpetual Russian option C is the solution of the linear program*

$$\begin{array}{ll} \textit{Maximize} & \langle C, \nu_\tau \rangle \\ \textit{Subj. to} & \langle f, \nu_\tau \rangle - \langle \widehat{A} f, \mu_0 \rangle - \langle Bf, \mu_1 \rangle = \langle f, \nu_0 \rangle, \quad \forall f \in \mathcal{D}, \\ & |\nu_\tau| \leq 1, |\mu_0| < \alpha^{-1}, |\mu_1| < \infty. \end{array}$$

7. Concluding Remarks

This paper illustrates how the pricing of options can be characterized by linear programs. European and American options are used to demonstrate the equivalence of the linear progam. Knock-out barrier options demonstrate how a change in the dynamics of the stock price process can be utilized for the characterization. The characterizations for knock-in and lookback options, on the other hand, indicate the benefits of including auxiliary processes whose dynamics have some sort of singularity. The perpetual Russian option adds to this by showing how time dependence through discounting is handled. The linear programs are infinite-dimensional and thus inherently of theoretical value.

These characterizations serve as a middle step on the road to option pricing. The challenge that remains is to use this approach as the basis for computation. The computation of perpetual Russian option prices has been accomplished by Helmes [**H**] using this type of characterization on an equivalent lower dimensional stopping problem; Shepp and Shiryaev [**SheShi**] relate the perpetual Russian option to this simpler stopping problem. The linear programming approach has also been used with success for computation of exit time problems ([**HRS**],[**HelS1**]) and stationary distributions ([**HelS2**]).

This paper has solely considered pricing of options in a complete market. The methods extend to incomplete markets where there are many equivalent martingale measures. These measures are characterized by the same constraints of the linear programs. Maximization will lead to the upper hedging price and minimization will lead to the lower hedging price. In such situations, a range of prices occurs and the challenge is then to determine *the* price of the option in some fashion. One approach is to optimize a criterion such as the mean-variance of a portfolio or the loss probability. (Control decisions enter such situations and the generality of Theorem 2.3 is needed.) This approach is natural for linear programming since the variables are measures and the values are determined by integration, a linear operation. Characterization of option prices in incomplete markets using linear programming will be the subject of a future paper.

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