7 Prime numbers (Continued)

Definition 7.1 *Strong Induction* Let $A_1, A_2, A_3, \dots, A_n, \dots$ be a sequence of statements. Suppose, we know that A_1 is true. Also, suppose that for each natural number $k > 1$ we know that "$A_1, \dots, A_{k-1}$ are true" implies "A_k is true" (i.e. all previous statements imply the next one). Then all statements $A_1, A_2, A_3, \dots, A_n, \dots$ are true.

Definition 7.2 The expression $n = p_1 p_2 \cdots p_k$, where each p_k is prime, is called a *prime factorization* of n. The prime factorization is *ordered* if $p_1 \le p_2 \le \cdots \le p_k$. Sometimes, by collecting equal p_i's we can shorten the notation to $n = p_1^{n_1} p_2^{n_2} ... p_m^{n_m}$, ($n_i \in \mathbb{N}$ for all i.)

Theorem 7.1 (The Fundamental Theorem of Arithmetic) *Each natural number $n > 1$ admits a unique ordered prime factorization.*

Problem 7.1 Prove by strong induction that for any $n \in \mathbb{N}$, $n > 1$ there exist a prime factorization of n.

Problem 7.2 Let $a, b \in \mathbb{N}$ and let p be a prime number. Prove that if $ab \,\vdots\, p$ then at least one of a, b is divisible by p. (Hint: Suppose $a \not\vdots\, p$, then use Problems 6.7 and 5.7.)

Problem 7.3 Prove the Fundamental Theorem of Arithmetic.

Problem 7.4 True or False: There are infinitely many prime numbers.

Problem 7.5 Describe $\gcd(a, b)$ in terms of the prime factorizations of a and b. (Hint: It's useful to think about the prime factorization of n as the infinite product $n = 2^{n_2} 3^{n_3} 5^{n_5} \dots p^{n_p} \dots$ over *all* primes, with $n_p = 0$ for missing p's.)

Definition 7.3 Let $a, b \in \mathbb{N}$. *The least common multiple* of a and b is the smallest natural number, which is divisible by both a and b:

$$\operatorname{lcm}(a, b) = \min\{m \in \mathbb{N} | m \,\vdots\, a \text{ and } m \,\vdots\, b\}$$

Problem 7.6 Find $\operatorname{lcm}(1, 2, 3, \dots, 20)$.

Problem 7.7 Prove $\gcd(a, b) \operatorname{lcm}(a, b) = ab$.