4 Division with remainder

Definition 4.1 Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. a is divided by b with remainder if $a = bq + r$ where $q, r \in \mathbb{Z}$ and $0 \leq r < b$. q and r are called *quotient* and *remainder* of division of a by b.

In order to make this terminology well-defined we need to prove the following fundamental theorem:

Theorem 4.1 *Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Then there exist unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $0 \leq r < b$.*

Problem 4.1 Fix $b \in \mathbb{N}$ and assume $a \geq 0$. Prove by induction on a the existence of $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $0 \leq r < b$.

Problem 4.2 Let $b \in \mathbb{N}$ and let $a \in \mathbb{Z}$ be negative. Prove the existence of $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $0 \leq r < b$.

Problem 4.3 Let $b \in \mathbb{N}$ and let $0 \leq r < b$. Prove that if $r \,\vdots\, b$ then $r = 0$.

Problem 4.4 Suppose $a = bq + r = bq' + r'$ with $q, q', r, r' \in \mathbb{Z}$, $0 \leq r < b$ and $0 \leq r' < b$. Prove that $r = r'$.

Problem 4.5 Prove Theorem 4.1.

In the following problems assume that a divided by c has the remainder r and b divided by c has the remainder s. In each problem make a table of the resulting remainder for $c = 3$ and $c = 4$.

Problem 4.6 Find the remainder of $a + b$ when divided by c.

Problem 4.7 Find the remainder of $a - b$ when divided by c.

Problem 4.8 Find the remainder of ab when divided by c.

Problem 4.9 Find the remainder of a^2 when divided by c.