Math 231 Sec 008 Instructor: M.Pryszczepko
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MIDTERM II SOLUTIONS

1. Let
$$f(x) = x^3 - 3x^2 + 5.$$
(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.

Solution: The domain of f is $D = (-\infty, \infty)$.
The derivative of f:
$$f'(x) = 3x^2 - 6x.$$
The critical numbers of f:
$$f'(x) = 0 \Leftrightarrow 3x^2 - 6x = 0 \Leftrightarrow 3x(x-2) = 0 \Leftrightarrow x = 0 \quad \text{or} \quad x = 2.$$
Since the graph of the derivative f' is a parabola that opens upward, (and we already know that 0 and 2 are the x-intercepts), we get:
$$f'(x) > 0 \Leftrightarrow (\, x < 0 \quad \text{or} \quad x > 2\,)$$
and
$$f'(x) < 0 \Leftrightarrow 0 < x < 2.$$
We are now ready to state the answers to both (a) and (b):
(a) The function f is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$, and decreasing on the interval $(0, 2)$. (We used the Increasing/Decreasing Test from page 240 of our textbook.)
(b) The function f has a local maximum at $x = 0$ and its value is $f(0) = 5$. The function f has a local minimum at $x = 2$ and its value is $f(2) = 2^3 - 3 \cdot 2^2 + 5 = 1$. (We used the First Derivative Test from page 241.)

2. Let
$$g'(x) = \frac{x}{(x^2+9)^2}.$$
(a) Find the intervals on which the graph of g is concave upward or downward.
(b) Find the first coordinates of the inflection points of the graph of g.

Solution: Since $(x^2+9)^2 \neq 0$ for all x, the domain of g' is $D_{g'} = (-\infty, \infty)$. The domain of g is $D_g = (-\infty, \infty)$. (Do you know why?)
The second derivative of g:
$$g''(x) = (g'(x))' = \frac{1 \cdot (x^2+9)^2 - 2(x^2+9) \cdot 2x \cdot x}{(x^2+9)^4} = \frac{(x^2+9)[(x^2+9) - 4x^2]}{(x^2+9)^4} = \frac{9-3x^2}{(x^2+9)^3}.$$
(We used both the quotient rule and the chain rule. When simplifying, factor first.)
The critical numbers of g':
$$g''(x) = 0 \Leftrightarrow \frac{9-3x^2}{(x^2+9)^3} = 0 \Leftrightarrow 9 - 3x^2 = 0 \Leftrightarrow 3(3-x^2) = 0 \Leftrightarrow x^2 = 3 \Leftrightarrow x = \pm\sqrt{3}.$$
We notice that the sign of the second derivative $g''(x) = (9-3x^2)/(x^2+9)^3$ is the same as the sign of its numerator $9 - 3x^2$ since $(x^2+9)^3 >$ for all x. To determine the sign of the numerator $9 - 3x^2$ we use its graph – a parabola that opens downward (and we already know that $\pm\sqrt{3}$ are the x-intercepts). So,
$$g''(x) > 0 \Leftrightarrow -\sqrt{3} < x < \sqrt{3}$$
and
$$g''(x) < 0 \Leftrightarrow (\, x < -\sqrt{3} \quad \text{or} \quad x > \sqrt{3}\,).$$

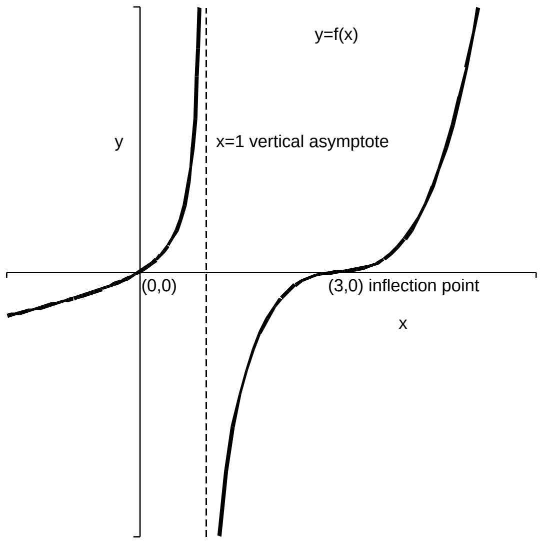
We are now ready to state the answers to both (a) and (b):

(a) The graph of g is concave upward on the interval $(-\sqrt{3}, \sqrt{3})$ and concave downward on the intervals $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. (We used the Concavity Test from page 243.)

(b) The first coordinates of the inflection points are $x = -\sqrt{3}$ and $x = \sqrt{3}$ since g is continuous at these points (do you know why?) and the graph of g changes the concavity there. (We used the definition of an inflection point from page 244.)

3. Sketch the graph of a function $y = f(x)$ that satisfies all of the given conditions:
(i) The domain of f is the set of all real numbers except $x = 1$.
(ii) $f'(x) > 0$ for all $x \neq 1$.
(iii) The line $x = 1$ is a vertical asymptote of the graph of f.
(iv) $f''(x) > 0$ if $x < 1$ or $x > 3$, and $f''(x) < 0$ if $1 < x < 3$.
(v) The graph of f has two x-intercepts: $x = 0$ and $x = 3$.

Solution: (ii) $\Rightarrow$ f is increasing on the intervals $(-\infty, 1)$ and $(1, \infty)$.
(iv) $\Rightarrow$ The graph of f is concave upward on the intervals $(-\infty, 1)$ and $(3, \infty)$, and concave downward on the interval $(1, 3)$.
(iii) and (ii) $\Rightarrow$ $\lim_{x \to 1^-} f(x) = \infty$ and $\lim_{x \to 1+} f(x) = -\infty$.
The graph of $y = f(x)$ is uniquely determined by the above information except for the behavior of $f(x)$ as $x \to -\infty$. We could have $\lim_{x \to -\infty} f(x) = b$ for some b (horizontal asymptote $y = b$) or $\lim_{x \to -\infty} f(x) = -\infty$ (slant asymptote or none at all).
Finally, using (i) and (v), we obtain the graph of $y = f(x)$:



4. A box with a square base and open top must have a volume of 32 ft^3. Use calculus methods to find the dimensions of the box that minimize its total surface area.

Solution: Let x be the length of the base and h the height of the box, both in feet. The volume of the box is

$$V = x^2 h$$

and the total surface area

$$A = x^2 + 4xh.$$

Since $V = 32$, we have

$$x^2 h = 32 \quad \Rightarrow \quad h = \frac{32}{x^2}.$$

Substituting in the formula for the area, we obtain

$$A(x) = x^2 + 4x\,\frac{32}{x^2} = x^2 + \frac{128}{x}.$$

We have to determine at what x the area function $A = A(x)$ assumes the absolute minimum on the domain $D = (0, \infty)$.
The derivative of A:

$$A'(x) = 2x - \frac{128}{x^2}.$$

The critical points of A:

$$A'(x) = 0 \Leftrightarrow 2x - \frac{128}{x^2} = 0 \Leftrightarrow 2x = \frac{128}{x^2} \Leftrightarrow x^3 = 64 \Leftrightarrow x = \sqrt[3]{64} \Leftrightarrow x = 4.$$

The second derivative of A:

$$A''(x) = (2x - 128x^{-2})' = 2 + 256x^{-3} = 2 + \frac{256}{x^3}.$$

So, $A''(4) = 2 + \frac{256}{4^3} > 0$ and therefore the function A has a local minimum at $x = 4$. (We used the Second Derivative Test from page 245.) It must be the absolute minimum because $x = 4$ is the only critical number on $D = (0, \infty)$. (See the First Derivative Test for Absolute Extreme Values on page 280.) The corresponding height is

$$h_{min} = \frac{32}{x_{min}^2} = \frac{32}{4^2} = 2.$$

Answer: The length of the base is $x_{min} = 4$ ft and the height of the box is $h_{min} = 2$ ft.

5. (a) State the Mean Value Theorem (MVT) by completing the following:
Let f be a function that satisfies the following hypotheses:
1. f is ... on the closed interval $[a, b]$.
2. f is ... on the open interval (a, b).
Then there is ... such that ...
(b) Does there exist a function f such that $f(-1) = 0$, $f(2) = 4$, and $f'(x) \leq 1$ for all x? (Hint: Suppose that there is such a function and use the MVT.)

Solution: (a) Let f be a function that satisfies the following hypotheses:
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. (See page 235.)

(b) Suppose that there is a function f satisfying the given conditions. Since $f'(x) < 1$ for all x, it follows in particular that f is differentiable and continuous on $(-\infty, \infty)$. So, f is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. The hypotheses of the MVT are satisfied for the function f on the interval $[-1, 2]$, therefore there is a number c in $(-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{4 - 0}{3} = \frac{4}{3} > 1.$$

But by assumption $f'(c) \leq 1$, a contradiction. It means that such a function does NOT exist.

6. An object moves in a straight line with the velocity

$$v(t) = 2t + 5.$$

Find the position function $s = s(t)$ of the object if you additionally know that $s(1) = 10$.

Solution: We know that $s'(t) = v(t)$, so

$$s(t) = \int v(t)\,dt = \int (2t + 5)\,dt = t^2 + 5t + C.$$

To find C, we use the initial condition:

$$s(1) = 10 \Rightarrow 1^2 + 5 \cdot 1 + C = 10 \Rightarrow C = 4.$$

So, $s(t) = t^2 + 5t + 4$.

7. (a) State the Fundamental Theorem of Calculus, Part 1 (FTC 1) by completing the following: If f is continuous on the interval $[a, b]$, then the function g defined by

$$g(x) = \ldots$$

is continuous on $[a, b]$ and differentiable on (a, b), and

$$g'(x) = \ldots$$

(b) Consider a function

$$S(x) = \int_0^x \sin(\pi t^2/2).$$

Find the smallest positive value of x at which the function S has a local extremum. Is it a local maximum or minimum? Justify your answer.

Solution: (a) If f is continuous on the interval $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$ and differentiable on (a, b), and

$$g'(x) = f(x).$$

(See page 342.)

(b) Let $f(x) = \sin(\pi x^2/2)$. Then $S(x) = \int_0^x f(t)\,dt$.

The derivative of S:

$$S'(x) = f(x) = \sin(\pi x^2/2) \qquad \text{by FTC 1.}$$

The critical numbers of S:

$$S'(x) = 0 \Leftrightarrow \sin(\pi x^2/2) = 0 \Leftrightarrow \frac{\pi x^2}{2} = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

The smallest positive solution will be the positive solution of the equation

$$\frac{\pi x^2}{2} = \pi \Leftrightarrow x^2 = 2 \Leftrightarrow x = \pm\sqrt{2}.$$

Hence $x = \sqrt{2}$.

To determine whether S has a local maximum or minimum at $x = \sqrt{2}$, we can use either the First or Second Derivative Test. Let's use the Second Derivative Test. We have

$$S''(x) = (\sin(\pi x^2/2))' = \pi x \cos(\pi x^2/2)$$

and

$$S''(\sqrt{2}) = \pi\sqrt{2}\cos(\pi(\sqrt{2})^2/2) = \pi\sqrt{2}\cos(\pi) = -\pi\sqrt{2} < 0.$$

So, S has the local maximum at $x = \sqrt{2}$.

8. (a) State the Fundamental Theorem of Calculus, Part 2 (FTC 2) by completing the following: If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \ldots$$

where F is a function such that ...
(b) Use the FTC 2 to evaluate the integral

$$\int_1^4 \frac{3}{\sqrt{x}}\,dx.$$

Solution: (a) If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is a function such that $F'(x) = f(x)$. (See page 344.)
(b) Using the derivative formula $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$, it is clear that $F(x) = 6\sqrt{x}$ is an antiderivative of the integrand $f(x) = \dfrac{3}{\sqrt{x}}$. (If you do not see it, check it.) So,

$$\int_1^4 \frac{3}{\sqrt{x}}\,dx = \Big[6\sqrt{x}\Big]_1^4 = 6\sqrt{4} - 6\sqrt{1} = 6.$$

9. (a) Evaluate the indefinite integral by making the given substitution:

$$\int \cos\theta \sin^4\theta\,d\theta, \qquad u = \sin\theta.$$

(b) Check your answer obtained in part (a) by differentiation. Write down each step very clearly.

Solution: (a) $u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$. So,

$$\int \cos\theta \sin^4\theta\,d\theta = \int u^4\,du = \frac{u^5}{5} + C = \frac{\sin^5\theta}{5} + C.$$

(b) Check:

$$\frac{d}{dx}\left(\frac{\sin^5\theta}{5}\right) = \frac{d}{dx}\left(\frac{1}{5}\sin^5\theta\right) = \frac{1}{5}\left(\frac{d}{dx}\sin^5\theta\right) = \frac{1}{5}\,5\sin^4\theta\left(\frac{d}{dx}\sin\theta\right) = \sin^4\theta\cos\theta.$$

Since $\sin^4\theta\cos\theta = \cos\theta\sin^4\theta$ is the integrand, the answer to part (a) is correct.

10. (a) State the definition of a definite integral by completing the following:
If f is a continuous function defined on $[a, b]$, we divide the interval $[a, b]$ into n subintervals of equal width

$$\Delta x = \ldots$$

We let $x_0, x_1, \ldots, x_n$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals. Then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \ldots$$

(b) Use the definition of a definite integral from part (a) with right endpoints as sample points to evaluate the integral

$$\int_0^2 x^2\,dx.$$

You will also need the following formula $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution: (a) If f is a continuous function defined on $[a,b]$, we divide the interval $[a,b]$ into n subintervals of equal width

$$\Delta x = \frac{b-a}{n}.$$

We let $x_0, x_1, \ldots, x_n$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals. Then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

(See page 326.)

(b) We have $f(x) = x^2$, $a = 0$, and $b = 2$. So, $\Delta x = (2-0)/n = 2/n$ and $x_i = a + i\Delta x = 2i/n$ $(i = 0, 1, \ldots, n)$. The right endpoints are $x_i^* = x_i$ $(i = 1, 2, \ldots, n)$. We can now evaluate the integral:

$$\begin{aligned}
\int_0^2 x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4i^2}{n^2}\,\frac{2}{n} \\
&= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} \\
&= \frac{4}{3}\lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \frac{4}{3}\lim_{n\to\infty} \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\
&= \frac{4}{3}\Big(1+0\Big)\Big(2+0\Big) \\
&= \frac{8}{3}.
\end{aligned}$$