On Some q-analogs of a Theorem of Kostant-Rallis

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1 Introduction

One of the deepest results in the theory of semi-simple groups is the Kostant-Rallis theorem which, in particular, is a multiplicity formula for the action of a reductive group on a graded module for the group. The purpose of this paper is to study several cases in which one can derive a graded multiplicity formula. The most notable success in this direction is Hesselink's graded version of Kostant's multiplicity formula for the action of a reductive group on the polynomials on the adjoint representation [Hes]. This case is included in our examples and our proof yields a slight simplification of his. In order to explain the context and results of this paper we need to develop some notation.

Let G denote a semi-simple linear algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ and let θ denote a regular involution with differential (also denoted) $\theta : \mathfrak{g} \to \mathfrak{g}$. Let K be the set of fixed points of θ in G and let $\mathfrak{k}$ denote the Lie algebra of K. As usual, we write

$$\mathfrak{p} = \{X \in \mathfrak{g} | \theta(X) = -X\}.$$

For any vector space V, let $\mathcal{P}(V)$ denote the complex valued polynomial functions on V. This ring is graded by degree, so let $\mathcal{P}^d(V)$ denote the subspace of homogeneous polynomials of degree d. We have,

$$\mathcal{P}(V) = \bigoplus_{d \geq 0} \mathcal{P}^d(V)$$

As usual, denote the subring of K-invariant polynomial functions on $\mathfrak{p}$ by $\mathcal{P}(\mathfrak{p})^K$. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$ (that is, a subspace of $\mathfrak{p}$ that is maximal subject to the condition that it is an abelian subalgebra of $\mathfrak{g}$ consisting of semi-simple elements). Let $M = \{k \in K | Ad(k)H = H, H \in \mathfrak{a}\}$. Set $M^* = \{k \in K | Ad(k)\mathfrak{a} \subset \mathfrak{a}\}$. We look upon M^*/M as a subgroup, $W(\mathfrak{a})$, of $GL(\mathfrak{a})$ (under the restriction of the adjoint action to $\mathfrak{a}$). Then $W(\mathfrak{a})$ is a Weyl group. We set $l = \dim \mathfrak{a}$ and we denote by $d_1, d_2, ..., d_l$ the degrees of the basic invariants of $W(\mathfrak{a})$. The Chevalley restriction theorem implies that $\mathcal{P}(\mathfrak{p})^K$ is a polynomial ring in homogeneous generators $u_1, ..., u_l$ with $\deg u_i = d_i$.

We set $\mathcal{I} = \mathcal{P}(\mathfrak{p})\mathcal{P}(\mathfrak{p})^K_+$and $\mathcal{I}^d = \mathcal{I} \cap \mathcal{P}^d(\mathfrak{p})$. So $\mathcal{I}$ is the ideal of $\mathcal{P}(\mathfrak{p})$ generated by the K-invariant functions which vanish at the origin and this ideal is graded by degree. $\mathcal{I}^d$ is stable under the action of K so it has a unique K-invariant complement, $\mathcal{H}^d$, in $\mathcal{P}^d(\mathfrak{p})$ (since K is reductive). We set $\mathcal{H} = \bigoplus_{d \geq 0} \mathcal{H}^d$. The total space $\mathcal{H}$ is the space of K-harmonic polynomials (that is the space annihilated by the K-invariant constant coefficient differential operators the annihilate the constants). The basic result of Kostant-Rallis says that

Theorem 1 *[K-R,cf. G-W]. As a $\mathcal{P}(\mathfrak{p})^K$-module $\mathcal{P}(\mathfrak{p})$ is free. That is, as a $\mathcal{P}(\mathfrak{p})^K$-module*

$$\mathcal{P}(\mathfrak{p}) \cong \mathcal{P}(\mathfrak{p})^K \otimes \mathcal{H}$$

and furthermore as a K-representation $\mathcal{H}$ is equivalent with the algebraically induced representation from M to K of the trivial representation.

In light of the theorem, Frobenius reciprocity implies that

$$\dim Hom_K(V, \mathcal{H}) = \dim V^{\rho(M)}$$

where (ρ, V) is an irreducible regular representation of K. We define the *graded multiplicity* of (ρ, V) in $\mathcal{H}$ to be $\sum_i q^i \dim Hom_K(V, \mathcal{H}^i)$. The purpose of this paper is to give explicit formulas for these graded multiplicities in several special cases.

We first note that this problem has been solved by [Hes], in the special case when $G = G_1 \times G_1$ with G_1 a semi-simple algebraic group and $\theta(x, y) = (y, x)$. We will give a slight simplification of Hesselink's original argument for this case in this paper. The analogue of the Kostant-Rallis theorem in this case is Kostant's famous decomposition of the adjoint representation of G_1 [K]. In section 2 we actually give a graded multiplicity formula that applies in a somewhat wider context (which includes the pairs

$(K, \mathfrak{p}) = (Sp_{2n}(\mathbb{C}), (\bigwedge^2 \mathbb{C}^{2n})/\mathbb{C})$, (F_4, V) with V the 26 dimensional irreducible representation). These results are based on a combinatorial formula (Proposition 2) for simple Lie algebras which applies to three cases: all roots, all short roots and all long roots. In the paper only the first two cases of the formula are used. It is hard to believe that such a beautiful relationship (in the case of the long roots) has no application. The rest of the paper will be devoted to the first non-trivial special case $(G = SL(4, \mathbb{C}), K = SO(4, \mathbb{C}))$ of the graded multiplicity that does not fit in the context of formulae of the type found in the next section.

The first named author wishes to thank Benedict Gross for having pointed his beautiful recent results related to the "short root representation" of a simple Lie group that is not simply laced. Those results inspired us to look at the context of the next section. We also thank David Meyer for explaining the basics of quantum computing to us and pointing out the importance of the material on $SL(4, \mathbb{C})$ in the context of the so-called "mixed case" of two qubits. These applications will be discussed in a forthcoming paper.

2 Some graded multiplicity Formulas.

In this section we will set up a general combinatorial framework that can be used to establish graded multiplicity formulas. Let H be a connected, simple, linear algebraic subgroup over $\mathbb{C}$. Fix, T, a maximal (algebraic) torus of H. Let Φ denote the root system of T acting on H. Choose a system, Φ^+, of positive roots in Φ and let Δ the simple roots in Φ^+. Let $W(H, T) = W$ be the Weyl group of T in H. If $\alpha \in \Phi$ then $s_\alpha \in W$ denotes the reflection about the hyperplane $\alpha = 0$. If $s \in W$ then set $Q_s = \{\alpha \in \Phi^+ | -s\alpha \in \Phi^+\}$. Let Ω be a subset of Φ that is W invariant and set $W_\Omega = \{s \in W | Q_s \subset \Omega\}$. Set $l = |\Omega \cap \Delta|$ (the cardinality of the set). Let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ (as usual). Set $\mathbf{D} = e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{s \in W} sgn(s) e^{s\rho}$.

Proposition 2 *W_Ω is the subgroup of W generated by the set*

$$\{s_\alpha | \alpha \in \Omega \cap \Delta\}.$$

Let $d_1, ..., d_l$ denote the degrees of a set of basic invariants of W_Ω in the space *spanned by* $\Omega \cap \Delta$. *Then*

$$\sum_{s\in W} sgn(s)\frac{e^{s\rho}}{\prod_{\alpha\in\Omega\cap\Phi^+}(1-qe^{s\alpha})} = \frac{\mathbf{D}\prod_{i=1}^{l}\frac{1-q^{d_i}}{1-q}}{\prod_{\alpha\in\Omega}(1-qe^{\alpha})}.$$

There are not many examples of subsets satisfying the assumptions that we have imposed on Ω. In fact, if H is simply laced then the only non-empty example is $\Omega = \Phi$. If Φ has two root lengths and if $\Omega \neq \Phi$ then it is the subset of roots of one of the two possible lengths. We now prove the first assertion of the proposition in light of this observation. Fix an invariant form $B(...,...)$ on $Lie(H)$ such that if $(...,...)$ is the corresponding dual form on $Lie(T)^*$ then $(\alpha,\alpha) > 0$ for $\alpha \in \Phi$. Let $\Phi_r = \{\alpha \in \Phi | (\alpha,\alpha) = r\}$. We assume (as we may) that $\Omega = \Phi_r$ for one of two possible values or $\Omega = \Phi$. We choose an enumeration $\{\alpha_1, ..., \alpha_n\}$ of Δ. For the sake of simplicity we will write s_i for s_{α_i}. Let $s \in W_\Omega$ and let $s = s_{i_1}\cdots s_{i_m}$ be a reduced expression for s. Then $\alpha_{i_m} \in Q_s \subset \Omega$. Since $Q_{ss_{i_m}} = s_{i_m}(Q_s - \{\alpha_{i_m}\})$, the first assertion of the proposition follows by induction on m. We also observe that $m = |Q_s| = l(s)$.

We will now prove the formula. If we multiply the left hand side of the formula by the denominator of the right hand side we obtain

$$\sum_{s\in W} sgn(s)e^{s\rho}\prod_{\alpha\in\Omega\cap\Phi^+}(1-qe^{-s\alpha}).$$

We may rewrite this expression as

$$\sum_{Q\subset\Omega\cap\Phi^+} q^{|Q|}(-1)^{|Q|}\sum_{s\in W} sgn(s)e^{s(\rho-<Q>)}$$

with $< Q >= \sum_{\alpha\in Q}\alpha$. The inner sum is 0 unless there exists $t \in W$ such that $\rho - < Q >= t\rho$. (this follows from the Weyl denominator formula cf. [G-W]) Thus Q must be Q_t for some t (cf.[G-W,7.3.7 p.331]) and thus $t \in W_\Omega$ and $|Q| = l(t)$. Since $sgn(s) = (-1)^{l(s)}$ we see that the expression above can be written as

$$\sum_{t\in W_\Omega} q^{l(t)}\sum_{s\in W} sgn(s)e^{s\rho}.$$

Now Chevalley [Chev] has shown that $\sum_{s\in W_\Omega} q^{l(s)} = \prod_{i=1}^{l}\frac{1-q^{d_i}}{1-q}$. The proposition now follows.

Set $\Omega^+ = \Omega \cap \Phi^+$. We define the q-analog of the Ω^+-partition function at μ, $\mathcal{P}_{\Omega^+}(q,\mu)$, to be the coefficient of e^μ in the expansion of

$$\frac{1}{\prod_{\alpha\in\Omega^+}(1-qe^\alpha)}.$$

If μ is Φ^+-dominant integral then we denote by χ_μ the character of the irreducible finite dimensional representation of (the simply connected covering group of) H with highest weight μ with respect to Φ^+ on T. Then the Weyl character formula says that

$$\mathbf{D}\chi_\mu = \sum_{s\in W} sgn(s)e^{s(\mu+\rho)}.$$

Let P_+ denote the set of dominant integral weights on T.

Corollary 3 *Let Ω be in the previous proposition. Then*

$$\frac{\prod_{i=1}^{l}\frac{1-q^{d_i}}{1-q}}{\prod_{\alpha\in\Omega}(1-qe^\alpha)} = \sum_{\mu\in P_+}\left(\sum_{s\in W} sgn(s)\mathcal{P}_{\Omega^+}(q,s(\mu+\rho)-\rho)\right)\chi_\mu.$$

We will write $\mathcal{P}(\xi) = \mathcal{P}_{\Omega^+}(\xi)$. The left hand side of the equation in the above proposition is given by

$$\sum_{s\in W} sgn(s)\sum_{\xi}\mathcal{P}(q.\xi)e^{s(\xi+\rho)}.$$

We rewrite this as

$$\sum_{s\in W} sgn(s)\sum_{\xi}\mathcal{P}(q,\xi-\rho)e^{s\xi}.$$

Since, for fixed ξ the sum over W is zero unless ξ is regular we see that this sum is equal to

$$\sum_{s\in W} sgn(s)\sum_{\xi\in P_+}\sum_{t\in W}\mathcal{P}(q,t\xi-\rho)e^{st\xi} = \sum_{\xi\in P_+}\left(\sum_{t\in W} sgn(t)\mathcal{P}(q,t\xi-\rho)\right)\sum_{s\in W} sgn(s)e^{s\xi}.$$

If $\xi \in P_+$ and the outer sum is non-zero then $\xi = \mu + \rho$ with $\mu \in P_+$. Thus the formula that we have been studying is

$$\sum_{\xi \in P_+} \left(\sum_{t \in W} sgn(t) \mathcal{P}(q, t(\xi + \rho) - \rho) \right) \sum_{s \in W} sgn(s) e^{s(\xi+\rho)}.$$

The result now follows from the Weyl character formula.

We will now show how the above combinatorial results imply graded multiplicity formulae. In the contexts that we will study we will only need the following easier version of the Kostant-Rallis theorem:

- $\mathcal{P}(\mathfrak{p})$ is free as a $\mathcal{P}(\mathfrak{p})^K$-module.

See [G-W,Lemma 12.4.14 p.569] for a proof of this result using only the Chevalley restriction theorem.

If G is a reductive algebraic group over $\mathbb{C}$ and if $V = \bigoplus_{i \geq 0} V^i$ is a graded G-module with $\dim V^i < \infty$ for all i then we write $ch_q(V)$ for the formal sum $\sum_{i \geq 0} q^i ch(V^i)$ where $ch(V^i)$ is the usual character of the G-module V^i.

As our first application if these ideas we look at the situation when $\Omega = \Phi$. If we take $G = H \times H$ and $\theta(x, y) = (y, x)$ then $K = \{(x, x) | x \in H\} \cong H$ and $\mathfrak{p} = \{(X, -X) | X \in Lie(H)\} \cong Lie(H)$ as an H-module under the adjoint representation. This is exactly the context of Kostant's theorem. In this case $\mathcal{P}_{\Phi^+}(q, \mu)$ is just the q-analog of the Kostant partition function as defined by Lusztig. We now apply the above results and obtain the following result of [Hes].

Theorem 4 *Let H be a connected, semi-simple algebraic group over $\mathbb{C}$ with $\mathfrak{h} = Lie(H)$. Fix a maximal (algebraic) torus, T, of H and a system of positive roots. If μ is a dominant integral character of T we denote by F^μ the irreducible finite dimensional representation of (of the simply connected covering group of) H with highest weight μ. Then*

$$\sum_i q^i \dim Hom_H(F^\mu, \mathcal{H}^i) = \sum_{s \in W} sgn(s) \mathcal{P}_{\Phi^+}(q, s(\mu + \rho) - \rho).$$

We note that since the weights of the adjoint representation consist of zero with multiplicity $l = \dim T$ union with Φ the q-character of the action of H on $\mathcal{P}(\mathfrak{h})$ restricted to T is given by

$$\frac{1}{(1-q)^l \prod_{\alpha \in \Phi} (1 - q e^\alpha)}.$$

Since the q-Hilbert series of $\mathcal{P}(\mathfrak{h})^H$ is $\frac{1}{\prod_{i=1}^{l}(1-q^{d_i})}$. The freeness assertion above implies that the q-character of $\mathcal{H}$ is

$$\frac{\prod_{i=1}^{l}\frac{1-q^{d_i}}{1-q}}{\prod_{\alpha\in\Phi}(1-qe^{\alpha})}.$$

The result now follows from the preceding corollary.

We now come to the examples inspired by some recent results of B. Gross. We assume that $\mathfrak{g}$ is simple and that if $\mathfrak{t}$ is the Lie algebra of a maximal torus T of K then the centralizer, $\mathfrak{a}$, of $\mathfrak{t}$ in $\mathfrak{p}$ is a Cartan subspace. The examples of this phenomenon are:

1. $G = SO(2n+2), K = SO(2n+1), \mathfrak{p} \cong \mathbb{C}^{2n+1}$.
2. $G = SL(2n, \mathbb{C})$ $(n > 1), K = Sp_{2n}(\mathbb{C}), \mathfrak{p} \cong (\bigwedge^2 \mathbb{C}^{2n})_0$ (orthogonal complement to the invariant).
3. $G = E_6, K = F_4, \mathfrak{p}$ the irreducible 26 dimensional representation of K.

In case 1 we take $l = 1$, for 3 we take $l = 2$ and for 2 we take $l = n-1$. If we replace H in the above discussion with K. Then we find that the weights of T on $\mathfrak{p}$ are 0 with multiplicity l combined with Ω the set of short roots for K with respect to T. We fix Φ^+, a system of positive roots for K with respect to T.

Theorem 5 *Let W be the Weyl group of K with respect to T. Let ρ be the half sum of the elements of Φ^+. Let Ω be the set of short roots in Φ and let $\Omega^+ = \Omega \cap \Phi^+$. We have $W_\Omega \cong W(\mathfrak{a})$. Furthermore, if μ is a dominant integral character of T and if F^μ is an irreducible regular representation of (the simply connected covering group of) K then*

$$\sum_i q^i \dim Hom_K(F^\mu, \mathcal{H}^i) = \sum_{s\in W} sgn(s)\mathcal{P}_{\Omega^+}(q, s(\mu+\rho)-\rho)$$

Once the first assertion has been established the formula is proved in exactly the same way as it was in the previous theorem. One can check that in case 1 we have $W(\mathfrak{a}) \cong S_2$, in case 3 we have $W(\mathfrak{a}) \cong S_3$ and in case 2 we have $W(\mathfrak{a}) \cong S_n$. An examination of the Dynkin diagrams and and an application of the first part of Proposition 2 implies that $W(\mathfrak{a}) \cong W_\Omega$.

In the work of B. Gross there is one other case that fit the pattern of his theory. As it turns out the methods of this section also apply to this case.

- $K = G_2, \mathfrak{p}$ the irreducible 7 dimensional representation of K.

In this case K is actually the fixed point group of an automorphism of order 3 of $SO(7, \mathbb{C})$. $\mathcal{P}(\mathfrak{p})^K$ is the polynomial ring in one variable (a degree 2 invariant). As before we take Ω to be the set of short roots. Here $\mathcal{H}^i$ is just the space of classical spherical harmonics in 7 variables of degree i. Thus the q-multiplicity formula amounts to a branching rule from $SO(7)$ to G_2.

Theorem 6 *Let W be the Weyl group of G_2 and let Ω denote the set of short roots. Let $\mathcal{H}^i$ denote the space of spherical harmonics in 7 variables homogeneous of degree i. Let F^μ denote a finite dimensional representation of G_2 with highest weight μ. Then*

$$\sum_i q^i \dim Hom_K(F^\mu, \mathcal{H}^i) = \sum_{s \in W} sgn(s) \mathcal{P}_{\Omega^+}(q, s(\mu + \rho) - \rho)$$

3 The case of (SL(4,$\mathbb{C}$),SO(4,$\mathbb{C}$)).

In this section we will contrast the beautiful examples of the previous section with the apparently simpler case of the title of this section. We look at $G = SL(4, \mathbb{C})$ and $\theta(g) = (g^T)^{-1}$ (here g^T is the usual transpose). So $K = SO(4, \mathbb{C})$. We observe that the quadratic polynomial on $M_2(\mathbb{C})$ given by $p(X) = \det X$ is invariant under the action of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ given by $(g, h)X = gXh^T$. Thus a dimension count shows that $SO(4, \mathbb{C})$ is isomorphic with the image of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ in $GL(M_2(\mathbb{C}))$ under this action. We will use this identification to parametrize the representations of $SO(4, \mathbb{C})$ by pairs of integers. Thus $F^{k.l}$ will denote the tensor product of the $k + 1$ dimensional irreducible representation of the $SL(2, \mathbb{C})$ in the first factor with the $l + 1$ dimensional $SL(2, (\mathbb{C})$ representation of the second factor. The standard action of $SO(4, \mathbb{C})$ on $\mathbb{C}^4$ is thus $F^{1,1}$. The representation on $\mathfrak{p}$ is $F^{2,2}$.

We fix the maximal torus, T,of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ that is the product of the diagonal matrices in each of the factors (each a maximal torus of $SL(2, \mathbb{C})$. If (π, V) is a finite dimensional representation of G then we write $char(V)$ for the character of V restricted to the maximal torus. The characters of the irreducible representations are given as follows. Let $t(s)$ denote the diagonal two by two matrix with entries s, s^{-1}. If F^k denotes a fixed choice of an irreducible $k + 1$-dimensional representation of $SL(2, \mathbb{C})$ then one has

$char(F^k)(t(s)) = \frac{(s^{k+1}-s^{-k-1})}{(s-s^{-1})}$. Hence if we write $\chi^k(s) = charF^k(t(s))$ we have

$$charF^{k,l}\left(t(s_1),t(s_2)\right) = \chi^k(s_1)\chi^l(s_2)$$
$$= \frac{(s_1^{k+1}-s_1^{-k-1})(s_2^{l+1}-s_2^{-l-1})}{(s_1-s_1^{-1})(s_2-s_2^{-1})}$$

Recall that if (π, V) is a representation of G with V a graded vector space $V = \bigoplus_{i\geq 0} V^i$ and each V^i a G-invariant and finite dimensional subspace then we will use the notation $char_q(V)$ for the formal sum $\sum_{i\geq 0} q^i char(V^i)$.

We therefore have $\mathfrak{k} = \mathfrak{s}l_2 \oplus \mathfrak{s}l_2 \cong F^{2,0} \oplus F^{0,2}$ and $\mathfrak{p} = \{X \in sl_4 | X^T = X\} \cong F^{2,2}$ as an $SO(4,\mathbb{C})$ representations. In this context the Kostant-Rallis Theorem says that, in this case,

$$\mathcal{P}(F^{2,2}) \cong \mathbb{C}[u_1,u_2,u_3] \otimes \mathcal{H}$$

with $u_i = Tr(X^{i+1})$ for $i = 1,2,3$ basic invariants and that $\mathcal{H}$ is equivalent with the representation of G induced from the trivial representation of the diagonal matrices, M, in (the usual realization of) $SO(4,\mathbb{C})$ (that is, M is isomorphic with the group of diagonal 4×4 matrices with diagonal entries ± 1 and determinant 1) . Thus Frobenius reciprocity allows us to compute the multiplicity, $m(k,l)$ of $F^{k.l}$ in $\mathcal{H}$. It follows that

$$\mathcal{H} = \bigoplus_{k,l\geq 0} m(k,l) F^{k,l}$$

with $m(k,l) = \dim(F^{k,l})^M$. Our problem is to compute the multiplicity, $m_d(k,l)$, of $F^{k,l}$ in $\mathcal{H}^d$ that is to say

$$\mathcal{H}^d = \bigoplus_{k,l\geq 0} m_d(k,l) F^{k,l}$$

A direct calculation which involves the calculation of the preimage of M in $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ yields

$$m(2k,2l) = \begin{cases} \frac{(2k+1)(2l+1)-3}{4} & \text{for } k-l \text{ odd,} \\ \frac{(2k+1)(2l+1)+3}{4} & \text{for } k-l \text{ even.} \end{cases}$$

while, $m(k,l) = 0$ if either k or l is odd. We will leave this not completely trivial calculation to the reader.

We now begin the calculation of q-characters in earnest. We note that

$$char_q(\mathcal{P}(F^{2,2})) = char_q\,\mathcal{P}(F^{2,2})^{SO(4,\mathbb{C})}\, char_q\,\mathcal{H}$$
$$= \frac{char_q\,\mathcal{H}}{(1-q^2)(1-q^3)(1-q^4)}$$

Also the characters of T are parametrized by pairs of integers as follows

$$(t(s_1), t(s_2)) \mapsto s_1^k s_2^l$$

hence the weights of $F^{2,2}$ are the $\chi^{k,l}$ with $k, l \in \{-2, 0, 2\}$. We therefore have (using $s = s_1, t = s_2$)

$$char_q\,\mathcal{P}(F^{2,2}) = \frac{1}{\prod_{i,j=-2,0,2}(1-qs^it^j)}$$

Hence, finding the graded multiplicity is equivalent to finding the polynomials $p_{k\,l}(q)$ such that,

$$\frac{(1-q^2)(1-q^3)(1-q^4)}{\prod_{i,j=-2,0,2}(1-qs^it^j)} = \sum_{k,l\geq 0} p_{k\,l}(q)\chi^{2k,2l}(s,t) \qquad (1)$$

Our initial approach to calculating the polynomials $p_{k,l}(q)$ involved a large number of computer calculations. Gradually we found experimental evidence for some beautiful patterns. Here is a list of the main ones that were observed.

- For all, $j \geq 0$

$$p_{j\,j}(q) = \frac{q^j(1-q^{j+2})(1-q^{j+1}) + q^{j+4}(1-q^j)(1-q^{j-1})}{(1-q)^2(1+q)}$$

- For all, $j \geq 0$

$$p_{j+1\,j}(q) = \frac{q^{j+2}(1-q^{j+2})(1-q^j)}{(1-q)^2}$$

But most importantly,

- (The shift formula) For all, $k, l \geq 0$ with $k \geq l$,

$$p_{k+2\,l}(q) - q^2 p_{k\,l}(q) = q^{2k-l+4}\left(\frac{1-q^{2l+1}}{1-q}\right)$$

It is clear that above three conditions completely characterize a set of polynomials $p_{k,l}(q)$ given as follows: If $k \geq l, k-l \in 2\mathbb{Z}$ then it must be given by

$$\frac{q^k(1+q^2+q^4) - q^{k+l}(q+q^2+q^3+q^4) + q^{2k}(q^{l+3} - q^{-l+2}) + q^{k+2l+3}}{(1-q)(1-q^2)} \tag{2}$$

and if $k \geq l, k-l \in 2\mathbb{Z}+1$ then it must be given by

$$\frac{(q^{2k+l} + q^{k+2l})(q^3 - q^4) + q^{2k-l}(q^3 - q^2) + q^k(q - q^4) + q^{k+l}(q^5 - q)}{(1-q^2)(1-q)^2} \tag{3}$$

We were fortunate to find that all of these "guesses" are correct. The second named author's thesis will contain an *a priori* proof of the shift formula in a slightly strengthened form which is sufficient to prove the above formulas. Our method of proof involves using geometric series to close the sum the formal series

$$\sum_{k,l \geq 0} p_{k\,l}(q) \chi^{2k,2l}(s,t)$$

and see that it is equal to $\frac{(1-q^2)(1-q^3)(1-q^4)}{\prod_{i,j=-2,0,2}(1-qs^it^j)}$.

This was carried out (with the aid of MAPLE although an enterprising reader could do it by hand) and the upshot is

Theorem 7 *The graded multiplicity of $F^{2k,2l}$ in $\mathcal{H}$ is given by the equations (2) and (3).*

4 $SO(4,\mathbb{C})$ invariants in $M_4(\mathbb{C})$

As an application of the previous theorem we will calculate the Hilbert series of $\mathcal{P}(M_4(\mathbb{C}))^{SO(4,\mathbb{C})}$. We first observe that

$$M_4(\mathbb{C}) \cong \mathfrak{p} \oplus \mathfrak{k} \oplus \mathbb{C}I$$

as $K = SO(4,\mathbb{C})$-representation. Here $\mathfrak{k} = Lie(K)$. If we have a graded decomposition of the polynomial functions on each of these irreducible components of $M_4(\mathbb{C})$, then a graded decomposition of $\mathcal{P}(M_4(\mathbb{C}))$ can be calculated by evaluating all representations that arise from tensoring irreducibles occurring in the space of polynomial functions on each of the components of $M_4(\mathbb{C})$. The Hilbert series for the invariants will then be the graded multiplicity of the trivial representation. To carry this program out in this case we first observe:

Proposition 8 *As a K-representation,*

$$char_q \mathcal{P}(\mathfrak{k}) = \frac{\sum_{k,l\geq 0} q^{k+l} \chi^{2k,2l}(s,t)}{(1-q^2)^2}$$

As an $SL(2,(\mathbb{C}) \times SL(2,(\mathbb{C})$-representation, $\mathfrak{k} \cong F^{2,0} \oplus F^{0,2}$ and as an $SL(2,\mathbb{C})$-representation,

$$char_q \mathcal{P}(F^2) = \frac{\sum_{k\geq 0} q^k \chi^{2k}(s)}{1-q^2}$$

This follows from the observation that $SL(2,\mathbb{C})$ is locally isomorphic with $SO(3,\mathbb{C})$ and, hence, the above decomposition follows from the classical theory of spherical harmonics. The result can also be verified directly by closing the sum to a rational expression and noting that it is equal to $\frac{1}{(1-qs^2)(1-qs^0)(1-qs^{-2})}$.

We next observe that $\mathcal{P}(F^{2,0} \oplus F^{0,2}) \cong \mathcal{P}(F^{2,0}) \otimes \mathcal{P}(F^{0,2})$ as an $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$-representation and hence

$$char_q \mathcal{P}(F^{2,0} \oplus F^{0,2}) = \frac{\sum_{k\geq 0} q^k \chi^{2k,0} \sum_{l\geq 0} q^l \chi^{0,2l}}{(1-q^2)^2}$$

Since

$$F^{n,0} \otimes F^{0,m} \cong F^{n,m}$$

as $SO(4,\mathbb{C})$ representations the result follows.

We now give the advertized Hilbert series

Theorem 9

$$char_q \mathcal{P}\ (M_4(\mathbb{C}))^{SO(4,\mathbb{C})} = \frac{q^{15}+q^{11}+q^{10}+3q^9+2q^8+2q^7+3q^6+q^5+q^4+1}{(1-q^4)^3(1-q^3)^2(1-q^6)(1-q^2)^3(1-q)}$$

We observe that the representation $F^{k,l} \bigotimes F^{r,s}$ has a $SO(4,\mathbb{C})$-invariant if and only if both equalities $k = r$ and $j = s$ hold. This implies that

$$char_q \mathcal{P}\ (M_4(\mathbb{C}))\ ^{SO(4,\mathbb{C})} = \frac{\sum_{k,l\geq 0} q^{k+l} p_{k\,l}(q)}{(1-q)(1-q^2)^3(1-q^3)(1-q^4)}$$

The latter expression can be summed formally by substituting our formulas for the $p_{kl}(q)$ closing the arising geometric series to obtain the rational function given in the statement. (The authors carried out this calculation with the aid of MAPLE.)

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