Math 623 Complex Analysis: First Assignment

You may not discuss these problems with anyone but me. Violators will be subject to any and all permissable academic sanctions.

1. Solve for z:
$$z^2 + (1-7i)z - (18+i) = 0,$$
and write your solutions in the standard form $a+bi$ where a and b are real. Check your answers.

2. Solve $z^6 + 1 = 0$ for z. Write all six solutions in the form $a+bi$ where a and b are real numbers.

3. Solve for z and write your answer in the form $a+bi$ where a and b are real numbers. Check your answer.
$$\frac{z+2-i}{z+3-4i} = 2-7i.$$

4. Draw the set $\{z \in C : (6+i)z + (6-i)\overline{z} + 5 = 0\}$ in the complex plane. What common geometric object have you graphed? How would one describe it without using complex numbers? Be as complete as possible.

5. Draw the set $\{z \in C : z\overline{z} + (1+i)z + (1-i)\overline{z} + 1 = 0\}$ in the complex plane. What common geometric object have you graphed? How would one describe it without using complex numbers? Be as complete as possible.

6. Consider the expression
$$f(z) = \frac{1}{z-\alpha}\left(\frac{1}{z^2} - \frac{1}{\alpha^2}\right)$$
where α is a fixed complex number. If we were to define a function on all the complex numbers by sending z to $f(z)$ when $z \neq \alpha$, how should define this function at α so that it is continuous at α? Justify your answer.

7. Use the identity
$$1 + z + \ldots + z^{n-1} = \frac{1-z^n}{1-z}, \quad z \neq 1$$
to show that for any complex numbers $\alpha \neq \beta$,
$$\sum_{k=0}^{n-1} \alpha^{n-1-k}\beta^k = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$
Use this to show that the polynomial $P(z) = z^n - \alpha^n$ is divisible by $z - \alpha$ when n is a positive integer. Conclude that for any non-constant polynomial $Q(z)$ and any complex number α, $Q(z) - Q(\alpha)$ is divisible by $z - \alpha$, so that if $Q(\alpha) = 0$ then $z - \alpha$ divides $Q(z)$. (See pages 7 and 8 of your textbook.)

8. Suppose that $|\alpha| \neq |\beta|$ and n is a positive integer. Show that
$$\left|\frac{\alpha^n - \beta^n}{\alpha - \beta}\right| \leq \frac{|\alpha|^n - |\beta|^n}{|\alpha| - |\beta|}$$

9. Let (z_1, z_2, z_3) and (w_1, w_2, w_3) be two triples of distinct complex numbers. Each triple determines a triangle by letting each number be a vertex. Show that these triangles are similiar (with vertex z_k corresponding to w_k) if and only if
$$\frac{z_1 - z_2}{w_1 - w_2} = \frac{z_2 - z_3}{w_2 - w_3} = \frac{z_3 - z_1}{w_3 - w_1}.$$
Hint: Think about Side-Side-Side and Angle-Angle-Angle.

10. For $z \neq 1$, define
$$f(z) = \frac{z+1}{z-1}.$$
Show that f maps $\{z \in C : Re(z) < 0\}$ into $\{w \in C : |w| < 1\}$ and vice versa. Extra credit: Show you can replace into with onto. Hint: Write $f(x+yi)$ in the form $u+vi$.