Transform Methods

Many theorems in probability deal with sums of independent random variables. Formulae for the distributions of such sums are frequently hard to deal with. Transform methods are frequently successful in such cases as they take advantage of two facts:

- $A^{b+c} = A^b A^c$
- The expected value of the product of independent random variables is the product of the expected values of those random variables.

1 Types of transforms

For the moment, let X be a real-valued random variable, and let z be a complex number. We can try to define the quantity $T(z)$ by $T(z) = \mathrm{E}[\exp(zX)]$ provided this expected value exists. Here are some special cases of interest.

1.1 Moment generating functions

If we restrict z to be a real number, the function $z \to \mathrm{E}[\exp(zX)]$ is called the **Moment Generating Function** of X. It is not at all clear that this function is defined for any value of z other than 0. For example, if X has the Cauchy distribution then

$$\mathrm{E}[\exp(zX)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\exp(zx)}{1+x^2} dx,$$

and this integral does not converge for any z except $z = 0$. On the other hand, if X has the standard normal distribution then

$$\begin{aligned} \mathrm{E}[\exp(zX)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(zx)\exp\left(-\frac{x^2}{2}\right)\,dx \\ &= \exp\left(\frac{z^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-z)^2}{2}\right)\,dx \\ &= \exp\left(\frac{z^2}{2}\right) \end{aligned}$$

for any real number z at all (and, for that matter, any complex number as well).

On the third hand, if X has an exponential distribution with rate 1, then

$$\begin{aligned} \mathrm{E}[\exp(zX)] &= \int_0^{\infty} \exp(zx)\exp(-x)\,dx \\ &= \int_0^{\infty} \exp(-(1-z)x)\,dx \\ &= \frac{1}{1-z} \end{aligned}$$

provided that $1 - z > 0$, for otherwise the integral will not converge.

We are not restricted to random variables with densities. For example, if X has a binomial distribution, then

$$\begin{aligned} \mathrm{E}[\exp(zX)] &= \sum_{n=0}^{N} \exp(zn) \binom{N}{n} p^n (1-p)^{N-n} \\ &= \sum_{n=0}^{N} \binom{N}{n} (p\exp(z))^n (1-p)^{N-n} \\ &= (p\exp(z) + (1-p))^N \end{aligned}$$

Moment generating functions get their name from the following property. Let $M(z) = \mathrm{E}[\exp(zX)]$ exist for all $z \in (-a, a)$ for some $a > 0$, and let $M^{(k)}$ denote the k^{th} derivative of M. Then $M^{(k)}(0) = \mathrm{E}[X^k]$. Ignoring some mathematical niceties for the moment, here is one way to see that this is true. Suppose that the Taylor series of M at 0 is equal to M itself. (This is a big if, and the heart of the matter mathematically.) In other words,

$$M(z) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} M^{(k)}(0) z^k.$$

On the other hand,

$$\begin{aligned} M(z) &= \mathrm{E}\left[1 + \sum_{k=1}^{\infty} \frac{1}{k!} z^k X^k\right] \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{E}[X^k] z^k. \end{aligned}$$

Since two power series agree if and only if the coefficient agree, we get what we are after.

For example, in the case of the binomial distribution, $M(z) = (p\exp(z) + (1-p))^N$, so $M'(z) = N(p\exp(z)+(1-p))^{N-1} p\exp(z)$ and $M'(0) = Np$, giving us the well-known fact that $\mathrm{E}[X] = Np$. In the case of the exponential distribution, we use the power series idea. Restrict z to be in $(-1, 1)$. Then

$$\begin{aligned} M(z) &= \frac{1}{1-z} \\ &= 1 + \sum_{k=1}^{\infty} z^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} k! z^k \end{aligned}$$

so $\mathrm{E}[X^k] = k!$.

Exercises

1. Calculate the moment generating function for the Poisson distribution.
2. Calculate the moment generating function for the geometric distribution.
3. The bilateral exponential density is given by $(1/2)\exp(-|x|)$ for all real numbers x. Show that the moment generating function of a random variable X with the bilateral exponential density is $1/(1-z^2)$ for $|z| < 1$. Use this to give a formula for $\mathrm{E}[X^{2n}]$ for positive integers n.
4. Find a formula for $\mathrm{E}[X^{2n}]$ for all positive integers n if X has the standard normal density.

2 Characteristic Function

A major drawback to the moment generating function is that its domain varies from random variable to random variable, and the condition that it exist on an open interval containing 0 is quite restrictive. As an alternative, recall that if θ is a real number then $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ satisfies $|\exp(i\theta)| = 1$. Hence for any random variable X at all, and any real number θ, $\mathrm{E}[\exp(i\theta X)]$ is always defined. The function $C(\theta) = \mathrm{E}[\exp(i\theta X)]$ is called the **characteristic function** of X. We have the following relation between the derivatives of C and the moments of X (k is a non-negative integer):

- If $\mathrm{E}[|X|^k] < \infty$ then $i^j\mathrm{E}[X^j] = C^{(j)}(0)$ for $j = 0, 1, \ldots, k$.
- If $C^{(2k)}(0)$ exists then $i^j\mathrm{E}[X^j] = C^{(j)}(0)$ for $j = 0, 1, \ldots, 2k$.

For now, the only characteristic function you need to know is for standard normal, where $C(\theta) = \exp(-\theta^2/2)$. Note that it is $M(i\theta)$ where M is the moment generating function!

It is important to note, however, that if X is an integer valued random variable, then its characteristic function is a Fourier series:

$$\begin{aligned} C(\theta) &= \sum_{k=-\infty}^{\infty} \exp(ik\theta)p_k \\ &= p_0 + \left(\sum_{k=1}^{\infty}(p_k + p_{-k})\cos(k\theta)\right) + \left(\sum_{k=1}^{\infty} i(p_k - p_{-k})\sin(k\theta)\right), \end{aligned}$$

where $p_k = \Pr(X = k)$. This is an important observation that we shall build on later. For now, note that

$$p_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(\theta)\exp(-k\theta)\; d\theta,$$

so that $C(\theta)$ encodes all the probability information of the distribution of X in the integer case. The same is true in general.