The relation between the Poisson, Binomial, and Negative Binomial Distributions

1 Binomial and Poisson Distributions

The Poisson distribution may be arrived at as a limiting case of the binomial distribution. Here are two ways to see this.

1.1 Direct Method

Let $t > 0$ be fixed, and consider a sequence of binomial mass functions $B_k(N, t/N)$ for $N > t$. That is,

$$B_k(N, t/N) = \left(\begin{array}{c} N \\ k \end{array} \right) \left(\frac{t}{N}\right)^k \left(1 - \frac{t}{N}\right)^{N-k}.$$

We are tossing a coin N times that has probability t/N of coming up heads and looking at the probability of getting k heads. By expanding terms and rearranging them we see that

$$B_k(N, t/N) = \frac{t^k}{k!} \left(1 - \frac{t}{N}\right)^N \frac{N(N-1)\ldots(N-(k-1))}{N^k} \left(1 - \frac{t}{N}\right)^{-k}.$$

As $N \to \infty$ the two rightmost terms in this expression converge to 1, and the second term from the left converges to e^{-t}, so we have

$$\lim_{N\to\infty} B_k(N, t/N) = \frac{t^k}{k!} e^{-t},$$

which we recognise as the Poisson distribution.

1.2 Characteristic Function Method

The characteristic function for the Binomial(N,t/N) distribution is

$$\begin{aligned} C_N(\theta) &= \sum_{k=0}^{N} e^{i\theta N} \left(\begin{array}{c} N \\ k \end{array} \right) \left(\frac{t}{N}\right)^k \left(1 - \frac{t}{N}\right)^{N-k} \\ &= \left(1 - \frac{t}{N} + \frac{te^{i\theta}}{N}\right)^N \\ &= \left(1 + \frac{t(e^{i\theta} - 1)}{N}\right)^N \end{aligned}$$

from which we see that

$$\lim_{N\to\infty} C_N(\theta) = \exp(t(e^{i\theta} - 1))$$

which is the characteristic function of the Poisson distribution with mean t. As we have noted earlier, the convergence of a sequence of characteristic functions to a characteristic function implies the convergence of the corresponding distributions to the distribution of the limit characteristic function.

2 Poisson and Negative Binomial Distributions

We will now show that at if N has a Poisson distribution with mean t that there is a sequence of independent, identically distributed random variables X_k, independent of N such that

$$Y = \sum_{k=1}^{N} X_k$$

has

$$\Pr(Y = k) = (-1)^k \binom{-t}{k} p^k (1-p)^t.$$

What is more, we will show how to discover the distribution of the X_k using probability generating functions.

2.1 Probability generating function method

Suppose that $P_X(z) = \mathrm{E}[z^{X_1}]$. Then

$$\begin{aligned} \mathrm{E}[z^Y] &= \Pr(N=0) + \sum_{k=1}^{\infty} \mathrm{E}[z^{X_1+\ldots+X_k}] \Pr(N=k) \\ &= \sum_{k=0}^{\infty} (P_X(z))^k \frac{t^k}{k!} e^{-t} \\ &= \exp(t(P(z)-1)). \end{aligned}$$

On the other hand, the probability generating function of the Negative Binomial distribution we are looking for is

$$\begin{aligned} \sum_{k=0}^{\infty} z^k (-1)^k \binom{-t}{k} p^k (1-p)^t &= \sum_{k=0}^{\infty} (-1)^k \binom{-t}{k} (pz)^k (1-p)^t \\ &= \left(\frac{1-p}{1-pz} \right)^t. \end{aligned}$$

The only way for these two probability generating functions to agree is for

$$\begin{aligned} P_X(z) &= 1 + \log(1-p) - \log(1-pz) \\ &= 1 + \log(1-p) + \sum_{n=1}^{\infty} \frac{p^n}{n} z^n. \end{aligned}$$

Well, this last is the probability generating function of the probability mass function p_n given by $p_0 = 1 + \log(1-p)$ and $p_n = p^n/n$ for $n = 1, 2, \ldots$. So we have shown that if

$$\Pr(X_k = n) = p_n$$

then

$$\Pr(Y = n) = (-1)^n \binom{-t}{n} p^n (1-p)^t.$$