Midterm I: Markov Chains and Related Topics

1, 20 points: Brewer's reliever Derrick Turnbow had a terrible second half of the season in 2006. Believing that he needed to mix his pitches better he consulted with Robert Adair, author of *The Physics of Baseball*. Adair suggested to Turnbow that he use a Markov chain model for choosing his next pitch. Specifically, he said:

- If you throw a fastball, three out of five times the next pitch should be a fastball. If it isn't a fastball it should be either a curve or a change up in the ratio of three to one.
- If you throw a change up, seventy five percent of the time the next pitch should be a fastball. If it isn't a fastball it should either be a curveball or a change-up in the ratio of four to one.
- If you throw a curveball, eighty percent of the time the next pitch should be a fastball. If it isn't a fastball there should be the same chance it is a curveball or a change-up.

Unbeknownst to Turnbow, Adair is a Cubs fan, and he tells the Cubs manager "Sweet Lou" Piniella of his scheme. Piniella wants to simplify this information for his batters by telling them that 75 percent of Turnbow's pitches will be fastballs, 15 percent will be curveballs, and 10 percent will be change-ups. Check to see if Piniella knows what he is talking about.

Sorry Lou, stick to baseball. Lou has not found the stationary distribution of Turnbow's Markov chain.

Order the states as

$$(\text{Fastball, Changeup, Curveball}).$$

Lou is asserting that $\pi = (3/4, 1/10, 3/20)$ is the stationary distribution. The transition matrix P is

$$P = \begin{bmatrix} 3/5 & 1/10 & 3/10 \\ 3/4 & 1/20 & 4/20 \\ 4/5 & 1/10 & 1/10 \end{bmatrix}.$$

It is easy to check that $\pi \neq \pi P$.

2, 20 points: Suppose that the offspring distribution in a branching process is $\Pr(X = 0) = p$ and $\Pr(X = 2) = (1 - p)$ What value of p gives a 50% chance that the population of the branching process becomes extinct?

The probability of extinction is the smallest positive root of

$$t = p + (1 - p)t^2.$$

The roots of this equation are 1 and $p/(1 - p)$. Since $1 \neq 1/2$ we want $p/(1 - p) = 1/2$, which gives $p = 1/3$.

3, 20 points: Suppose that Fred begins a game of gambler's ruin with 10 euros against Ethel who has 20 euros. To make the game fair, the Ethel offers to play the game with a coin that comes up heads 2 out of 3 times on average, and that Fred should win one euro if the coin comes up heads. Set up a difference equation for f_x, Fred's chance of winning when he has x euros, and solve for f_x. What is Fred's probability of winning Ethel's 20 euros?

We have $f_0 = 0$, $f_{30} = 1$ and $f_x = (2/3)f_{x+1} + (1/3)f_{x-1}$ if $x \in \{1, 2, \ldots, 29\}$. If we try a solution of the form $f_x = s^x$ then we see that

$$s^x = (2/3)s^{x+1} + (1/3)s^{x-1},$$

so $s = (2/3)s^2 + (1/3)$. This gives $s = 1$ or $s = 1/2$. Thus

$$f_x = A + B(1/2)^x$$

for an appropriate choice of A and B. Since $f_0 = 0$ we must have $B = -A$, so

$$f_x = A(1 - (1/2)^x).$$

Since $f_{30} = 1$ we must have

$$f_x = \frac{1 - (1/2)^x}{1 - 2^{30}}$$

Ethel has been far too generous, as Fred's chances of winning her 20 euros are

$$f_{10} = \frac{1 - (1/2)^{10}}{1 - (1/2)^{30}} \approx 0.999.$$

4, 20 points: Having won Ethel's 20 euros, Fred wants to play a quick game of gambler's ruin with Ricky. They agree to use a fair coin, and to make the game quick, they each start with 2 euros. Set up a transition matrix for the Markov chain that models X_n, the amount of money Fred has in the game after n tosses of the coin. Classify the states and compute the expected amount of time that each player has 2 euros.

Since we know that 0 and 4 are absorbing states and $\{1, 2, 3\}$ is the single class of transient states, let us order the states as (0,4,2,1,3), so the transition matrix P has the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

(Other orderings of the states give other forms of P. However, this order reflects the symmetry in the problem better than the standard order!)

Since

$$P = \begin{bmatrix} I & 0 \\ B & Q \end{bmatrix}$$

where

$$Q = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

and that the entries of the first row of $(I-Q)^{-1}$ are the expected numbers of visits to the states $2,1,3$ starting from state 2 We have

$$I - Q = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

so

$$(I-Q)^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3/2 & 1/2 \\ 1 & 1/2 & 3/2 \end{bmatrix}$$

so the expected amount of time that they are tied is 2 plays of the game.

Or, since we only want $(I-Q)^{-1}_{1,1}$ we could use the cofactor representation of the inverse to compute this one entry directly.

5, 20 points: In a given year a law firm considers N claims, where N has a Poisson distribution with mean 1000. Each case has a 75% chance of being settled out of court and a 25% chance of going to trial. What is the probability generating function for the distribution of T, the number of cases that go to trial if we assume that all events are independent of one another? Use this to compute the mean and variance of T.

Let $C_k = 1$ if the k^{th} case goes to trial and 0 if it does not. Then the probability generating function of C_k is $\mathrm{E}[t^{C_k}] = (3/4) + (1/4)^t$. Therefore the probability generating function of T is

$$\begin{aligned} P(t) &= \mathrm{E}[t^T] \\ &= \Pr(N=0) + \sum_{n=1}^{\infty} \mathrm{E}[t^{C_1+\ldots+C_n}] \Pr(N=n) \\ &= \sum_{n=0}^{\infty} \left(\frac{3}{4} + \frac{t}{4}\right)^n \frac{1000^n}{n!} e^{-1000} \\ &= e^{250t-250} \end{aligned}$$

Then $\mathrm{E}[T] = P'(1) = 250$, $\mathrm{E}[T^2 - T] = P''(1) = (250^2)$, so $\mathrm{Var}(T) = 250^2 + 250 - 250^2 = 250$, or recognize that P is the probability generating function of a Poisson distribution with mean 250, so the mean and variance of T are both 250.