Midterm II Solutions: Continuous Time Markov Chains and Poisson Processes

Paper and pencil/pen only. Show your work other than simple calculus and arithmetic.

1: The time it takes Ricky to mow his lawn is exponentially distributed with mean 1 hours, and the time it takes Fred to mow his lawn is exponentially distributed with mean 2 hours. If Fred starts an hour before Ricky, what is the probability that Fred finishes before Ricky finishes?

Let R be the time it takes Ricky, and let F be the time it takes Fred. For $t > 0$ we have $\Pr(R > t) = \exp(-t)$ and $\Pr(F > t) = \exp(-t/2)$. Let A be the event that Fred finishes first. Then

$$\begin{aligned}
\Pr(A) &= \Pr(A \cap \{F \leq 1\}) + \Pr(A \cap \{F > 1\}) \\
&= \Pr(\{F \leq 1\}) + \Pr(A \cap \{F > 1\}) \\
&= \Pr(\{F \leq 1\}) + \Pr(F < R)\Pr(F > 1) \\
&= 1 - \exp(-1/2) + \frac{1}{3}\exp(-1/2) \\
&= 1 - \frac{2\exp(-1/2)}{3}.
\end{aligned}$$

Note that by the memoryless property,

$$\Pr(A \cap \{F > 1\}) = \Pr(F < R)\Pr(F > 1).$$

2: Give the definition of a Poisson Process in terms of properties of its increments.

A Poisson Process is a set of random variables $\{N_t, t \geq 0\}$ satisfying

1. $\Pr(N_0 = 0) = 1$;
2. $\{N_t, t \geq 0\}$ has the independent increments property;
3. There is a positive constant r such that for all $t \geq 0$ and $s > 0$, and each non-negative integer k,

$$\Pr(N_{t+s} - N_t = k) = \frac{(rs)^k}{k!}\exp(-rs).$$

3: Consider a Poisson process with rate 6. Show that the conditional distribution for N_3 given that $N_7 = 5$ is binomial.

Since $\Pr(0 \leq N_3 \leq N_5) = 1$, if $x \in \{0, 1, 2, 3, 4, 5\}^c$ then

$$\Pr(N_3 = x | N_7 = 5) = 0.$$

For the remaining values of x,

$$\begin{aligned}
\Pr(N_3 = x | N_7 = 5) &= \frac{\Pr(N_3 = x, N_7 = 5)}{\Pr(N_7 = 5)} \\
&= \frac{\Pr(N_3 - N_0 = x, N_7 - N_3 = 5 - x)}{\Pr(N_7 - N_0 = 5)} \\
&= \frac{\Pr(N_3 - N_0 = x)\Pr(N_7 - N_3 = 5 - x)}{\Pr(N_7 - N_0 = 5)}
\end{aligned}$$

We have

$$\begin{aligned}
\Pr(N_3 - N_0 = x) &= \frac{18^x}{x!}\exp(-18) \\
\Pr(N_7 - N_3 = 5 - x) &= \frac{24^{5-x}}{(5-x)!}\exp(-24) \\
\Pr(N_7 - N_0 = 5) &= \frac{42^5}{5!}\exp(-42)
\end{aligned}$$

so

$$\begin{aligned}
\Pr(N_3 = x | N_7 = 5) &= \frac{18^x \exp(-18) 24^{5-x} \exp(-24) 5!}{x!(5-x)! 42^5 \exp(-42)} \\
&= \frac{18^x 24^{5-x} 5!}{x!(5-x)! 42^5} \\
&= \begin{pmatrix} 5 \\ x \end{pmatrix} \left(\frac{3}{7}\right)^x \left(\frac{4}{7}\right)^{5-x}
\end{aligned}$$

which is the probability mass function for a binomial distribution.

4: Suppose that cars arrive a gas station at the rate of 15 per hour, but that the station can accommodate 4 or fewer cars at one time. If cars can only be serviced one at time at a rate of 12 per hour, what fraction of the time are four cars at the station?

We will model this as a birth and death process on $\{0, 1, 2, 3, 4\}$. The rate matrix is

$$\Lambda = \begin{bmatrix} -15 & 15 & 0 & 0 & 0 \\ 12 & -27 & 15 & 0 & 0 \\ 0 & 12 & -27 & 15 & 0 \\ 0 & 0 & 12 & -27 & 15 \\ 0 & 0 & 0 & 12 & -12 \end{bmatrix}$$

We need to solve $\vec{\pi}\Lambda = \vec{0}$ where $\vec{\pi} = [\pi_0, \pi_1, \pi_2, \pi_3, \pi_4]$ and π_4 is the quantity we are seeking.

We see that

$$\begin{aligned}
\pi_0 &= \pi_0 \\
\pi_1 &= \frac{5}{4}\pi_0 \\
\pi_2 &= \frac{5}{4}\pi_1 = \left(\frac{5}{4}\right)^2 \pi_0 \\
\pi_3 &= \frac{5}{4}\pi_2 = \left(\frac{5}{4}\right)^3 \pi_0 \\
\pi_4 &= \frac{5}{4}\pi_3 = \left(\frac{5}{4}\right)^4 \pi_0
\end{aligned}$$

so

$$\begin{aligned}
1 &= \pi_0 \left(1 + \left(\frac{5}{4}\right) + \left(\frac{5}{4}\right)^2 + \left(\frac{5}{4}\right)^3 + \left(\frac{5}{4}\right)^4\right) \\
&= \pi_0 \frac{(5/4)^5 - 1}{(5/4) - 1}.
\end{aligned}$$

Therefore

$$\pi_4 = \frac{(5/4)^4}{4((5/4)^5 - 1)}.$$

5: Consider a Birth and Death process X_t on $S := \{0, 1, 2, 3, 4, 5\}$ with birth rate $\lambda(s) = (5-s)^3$ and death rate $\delta(s) = 15s^2 - s^3$, and initial distribution $\Pr(X_0 = s) = s/15$ for $s \in S$. Derive and solve a differential equation for $\mathrm{E}[X_t]$.

First, $\mathrm{E}[X_0] = (0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2)/15 = 11/3$.

Next, if s is a state, then

$$\begin{aligned} \mathrm{E}[X_{t+h}|X_t = s] &= (s+1)(\lambda(s)h + o(h)) \\ &\quad + s(1 - \lambda(s)h - \delta(s) + o(h)) \\ &\quad + (s-1)(\delta(s)h + o(h)) \\ &\quad + o(h) \\ &= s + (\lambda(s) - \delta(s))h + o(h) \end{aligned}$$

Now, since

$$(5-s)^3 = 125 - 75s + 15s^2 - s^3$$

we have

$$\lambda(s) - \delta(s) = 125 - 75s$$

so

$$\mathrm{E}[X_{t+h}|X_t = s] = s + (125 - 75s)h + o(h)$$

Therefore, setting $M(t) = \mathrm{E}[X_t]$ we have

$$\begin{aligned} M(t+h) &= \sum_{s=0}^{5} \mathrm{E}[X_{t+h}|X_t = s] \Pr(X_t = s) \\ &= \sum_{s=0}^{5} (s + (125 - 75s)h + o(h)) \Pr(X_t = s) \\ &= M(t) + (125 - 75M(t))h + o(h) \\ \frac{M(t+h) - M(t)}{h} &= 125 - 75M(t) + \frac{o(h)}{h} \\ M'(t) &= 125 - 75M(t) \\ M'(t) + 75M(t) &= 125 \end{aligned}$$

To solve this differential equation we multiply the integrating factor $\exp(75t)$ to get

$$\begin{aligned} (\exp(75t)M(t))' &= 125\exp(75t) \\ \exp(75t)M(t) &= \frac{5}{3}\exp(75t) + C \\ M(t) &= \frac{5}{3} + C\exp(-75t) \\ M(t) &= \frac{5}{3} + 2\exp(-75t) \end{aligned}$$

since $M(0) = 11/3$.

6: Claims arrive at insurance company at the rate of D per day. The time it takes to process a claim is modeled by a random variable with mean μ and variance σ^2 and the number of claims is to be modeled by a Poisson process.

Assume the times to process claims are independent of each other and independent of the number of claims. Let X denote the number of hours needed to process five days worth of claims.

Derive the formula $\mathrm{Var}[X] = 5D(\sigma^2 + \mu^2)$ from first principles. You may assume that $\mathrm{E}[X] = 5D\mu$.

Let N be the number of claims in 5 days. N has a Poisson distribution with mean and variance equal to $5D$. We have

$$X = \sum_{j=1}^{N} X_j$$

with the understanding that $X = 0$ if $N = 0$.

$$\begin{aligned}
\mathrm{E}[X^2] &= \sum_{k=0}^{\infty} \mathrm{E}[X^2 \cdot I_{N=k}] \\
&= \sum_{k=1}^{\infty} \mathrm{E}[X^2 \cdot I_{N=k}] \\
&= \sum_{k=1}^{\infty} \mathrm{E}\left[\left(\sum_{j=1}^{N} X_j\right)^2 \cdot I_{N=k}\right] \\
&= \sum_{k=1}^{\infty} \mathrm{E}\left[\left(\sum_{j=1}^{k} X_j\right)^2 \cdot I_{N=k}\right] \\
&= \sum_{k=1}^{\infty} \mathrm{E}\left[\left(\sum_{j=1}^{k} X_j\right)^2\right] \mathrm{E}[I_{N=k}] \\
&= \sum_{k=1}^{\infty} (k\sigma^2 + k^2\mu^2)\,\mathrm{Pr}(N = k) \\
&= \sigma^2 \mathrm{E}[N] + \mu^2(\mathrm{Var}(N) + (\mathrm{E}[N])^2) \\
&= \sigma^2 \cdot 5D + \mu^2(5D + 25D^2).
\end{aligned}$$

Therefore

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = \sigma^2 \cdot 5D + \mu^2(5D + 25D^2) - 25D^2\mu^2 = 5D(\sigma^2 + \mu^2)$$

as desired.