Branching Processes

Before we begin our study of Branching Processes we will look at random sums.

1 Random Sums

Suppose that N is a random variable taking values in the non-negative integers. For example, N might have the Poisson distribution. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables, and suppose that these random variables are independent of N. We wish to study the random variable Y defined by

$$Y = \begin{cases} 0 & \text{if} \quad N = 0 \\ \sum_{k=1}^{N} X_k & \text{if} \quad N > 0 \end{cases}$$

We will show how to derive the mean, variance, characteristic function and moment generating function of Y in terms of the means, variances, characteristic functions and moment generating functions of N and the X_k.

In what follows $I_k = 1$ if $N = k$ and 0 otherwise. In particular, $\mathrm{E}[I_k] = \Pr(N = k)$ and I_k is independent of all the X_k.

1.1 Mean value

$$\begin{aligned} \mathrm{E}[Y] &= \sum_{k=0}^{\infty} \mathrm{E}[Y I_k] \\ &= \sum_{k=1}^{\infty} \mathrm{E}[(X_1 + \cdots + X_N) I_k] \\ &= \sum_{k=1}^{\infty} \mathrm{E}[(X_1 + \cdots + X_k) I_k] \\ &= \sum_{k=1}^{\infty} k\mathrm{E}[X_1]\mathrm{E}[I_k] \\ &= E[X_1] \sum_{k=1}^{\infty} k \Pr(N = k) \\ &= \mathrm{E}[X_1]\mathrm{E}[N] \end{aligned}$$

1.2 Variance

Here we make use of the fact that $\mathrm{Var}[Y] = \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2$ and that the variance of a sum is the sum of the variances if the summands are independent.

$$\begin{aligned} \mathrm{E}[Y^2] &= \sum_{k=1}^{\infty} \mathrm{E}[(X_1 + \cdots + X_N)^2 I_k] \\ &= \sum_{k=1}^{\infty} \mathrm{E}[(X_1 + \cdots + X_k)^2 I_k] \\ &= \sum_{k=1}^{\infty} \mathrm{E}[(X_1 + \cdots + X_k)^2] \Pr(N = k) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \left(\mathrm{Var}[X_1 + \cdots + X_k] + (k\mathrm{E}[X_1])^2 \right) \Pr(N = k) \\
&= \sum_{k=1}^{\infty} \left(k\mathrm{Var}[X_1] + (k\mathrm{E}[X_1])^2 \right) \Pr(N = k) \\
&= \mathrm{E}[N]\mathrm{Var}[X_1] + (\mathrm{E}[X_1])^2 \mathrm{E}[N^2]
\end{aligned}
$$

so

$$
\begin{aligned}
\mathrm{Var}[Y] &= \mathrm{E}[N]\mathrm{Var}[X_1] + \mathrm{E}[N^2](\mathrm{E}[X_1])^2 - (\mathrm{E}[N]\mathrm{E}[X_1])^2 \\
&= \mathrm{E}[N]\mathrm{Var}[X_1] + (\mathrm{E}[X_1])^2 \mathrm{Var}[N]
\end{aligned}
$$

1.3 Probability Generating Function

Suppose that z is a complex number and that $|z| \leq 1$. Define the function $P_N(z)$ by

$$
P_N(z) = \mathrm{E}[z^N] = \sum_{k=0}^{\infty} z^k \Pr(N = k)
$$

P_N is called the **probability generating function** of N. We looked at such functions when we considered the probability of first return to a state by a Markov chain. Taylor's theorem tells us that we can recover $k! \Pr(N = k)$ from P_N by differentiating P_N k times and evaluating the result at 0. We also can observe that

$$
\mathrm{E}[N] = P_N'(1)
$$

and

$$
\mathrm{E}[N(N-1)] = P''(1).
$$

Suppose now that the X_k also take values only in the non-negative integers, and let P_X denote their probability generating function. Then we can express the probability generating function of Y in terms of P_N and P_X:

$$
\begin{aligned}
\mathrm{E}[z^Y] &= \sum_{k=0}^{\infty} \mathrm{E}[z^Y I_k] \\
&= \mathrm{E}[z^0 I_0] + \sum_{k=1}^{\infty} \mathrm{E}[z^{X_1 + \cdots + X_N} I_k] \\
&= \Pr(N = 0) + \sum_{k=1}^{\infty} \mathrm{E}[z^{X_1 + \cdots + X_k} I_k] \\
&= \Pr(N = 0) + \sum_{k=1}^{\infty} \mathrm{E}[z^{X_1}]^k \Pr(N = k) \\
&= \Pr(N = 0) + \sum_{k=1}^{\infty} (P_X(z))^k \Pr(N = k) \\
&= P_N(P_X(z))
\end{aligned}
$$