# Additional one point problems for assignment 5

Newton's binomial theorem states that if $A$ is any real number and $|x| < 1$ then

$$(1+x)^A = 1 + \sum_{k=1}^{\infty} \binom{A}{k} x^k$$

where

$$\begin{aligned} \binom{A}{1} &= A \\ \binom{A}{k} &= \frac{A(A-1)\cdots(A-(k-1))}{k!} \quad \text{if } \ k > 1 \end{aligned}$$

If we consider the special case where $x = -p$, $0 < p < 1$ and $A = -B < 0$ then Newton's binomial theorem gives

$$1 = (1-p)^B + \sum_{k=1}^{\infty} (-1)^k \binom{-B}{k} p^k (1-p)^B$$

Hence for each $B > 0$ and $p \in (0,1)$ we can define a probability mass function on the non-negative integers, call it $q_k(p, B)$ by the rule

$$q_k(p,B) = \begin{cases} (1-p)^B & \text{if } \ k = 0 \\ (-1)^k \binom{-B}{k} p^k (1-p)^B & \text{if } \ k > 0 \end{cases}$$

This family of probability mass functions are called **negative binomial distributions**. Here are some exercises for you to familiarize you with this important family of distributions.

1. Show that if $B = 1$ we get the geometric distribution.

2. Show that the probability generating function for this family is

$$P(t) = \left(\frac{1-p}{1-pt}\right)^B.$$

3. Use the fact that if $P(t)$ is the probability generating function of $X$ then $P'(1) = \mathrm{E}[X]$ and $P''(1) = \mathrm{E}[X(X-1)]$ to show that the mean and variance for the negative binomial distribution are $Bp/(1-p)$ and $Bp/(1-p)^2$ respectively.

4. In the negative binomial distribution express the parameters $p$ and $B$ in terms of the mean and variance. Can any mean and variance be accounted for by a negative binomial distribution? Explain.

5. Use probability generating functions to show that if $T_1, T_2, \ldots, T_N$ are independent and identically distributed geometric random variables then $T_1 + \ldots + T_N$ has a negative binomial distribution.

6. Suppose that the offspring distribution of a branching process has a negative binomial distribution with $B = 2$. Find the probability of eventual extinction as a function of $p$ and graph it for $p \in (0,1)$.

7. Suppose that the offspring distribution of a branching process has a negative binomial distribution with $B = 1/2$. Find the probability of eventual extinction as a function of $p$ and graph it for $p \in (0, 1)$.

8. An insurance company believes that the number of claims it processes in a year follows a negative binomial distribution with $p = 1/3$ and $B = 20000$, and that the value of each claim has a Poisson distribution with mean 10000. What is the mean and variance of the amount the company pays out in a year if they assume that the claim amounts are independent of each other and of the number of claims?

9. An insurance company believes that the number of claims it processes in a year follows a Poisson distribution with mean 10000, and that the value of each claim has a negative binomial distribution with $B = 20000$ and $p = 1/3$. What is the mean and variance of the amount the company pays out in a year if they assume that the claim amounts are independent of each other and of the number of claims? Compare your answers to the previous problem. How are they the same, how are the different?