Stationary Poisson Processes

A counting process is a set of non-negative integer-valued random variables, $\{N_t, t \in [0, \infty)\}$ with three properties:

- $N_0 = 0$
- If $0 \le s \le t$ then $N_s \le N_t$;
- If $t \ge 0$ then $\lim_{s \to t^+} N_s = N_t$.

We will now consider a special sort of counting process called a **Poisson process**. A Poisson process is a counting process with these two additional properties:

- There is a constant $r > 0$, called the rate, so that if $0 \le s < t$ then

$$\Pr(N_t - N_s = k) = \frac{r^k (t-s)^k}{k!} e^{-r(t-s)}.$$

- For any integer m greater than 1, if $0 \le t_1 < t_2 \cdots < t_m$ the random variables $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots, N_{t_{m-1}} - N_{t_m}$ are mutually independent.

In the next sections we shall see how Poisson processes might arise.

1 Poisson processes and waiting times

Suppose that $T_1, T_2, \ldots$ is an infinite sequence of exponential functions with common survival function $S(t) = e^{-rt}$ for some $r > 0$. We view the T_k as the time between occurences of some type of event, such as the replacement of an electrical component, the arrival of the next customer at a bank, or the submission of the next claim to an insurance company. Let N_t be the number of events that have been observed by time t. It is easy to see that the following two events are the same:

$$\{N_t = 0\} = \{T_1 > t\}.$$

Now consider the events $\{N_t \le k\}$ and $\{T_1 + \ldots + T_{k+1} > t\}$. Our description of N_t is consistent with

$$\{N_t \le k\} = \{T_1 + \ldots + T_{k+1} > t\}. \tag{1}$$

For, if there have been no more than k events by time t, then the time of the $k+1^{\text{st}}$ event must be after time t, and conversely, if the time of the $k+1^{\text{st}}$ event is after time t, then the number of events by time t is less than or equal to k. Hece we define the N_t by (1). This tells us that if $k \ge 1$ then

$$\begin{aligned}
\Pr(N_t \le k) &= \Pr(T_1 + \ldots + T_{k+1} > t) \\
&= \int_t^\infty \frac{1}{k!}(ru)^k e^{-ru} r \, du && (2) \\
&= \frac{(rt)^k}{k!} e^{-rt} + \int_t^\infty \frac{1}{(k-1)!}(ru)^{k-1} e^{-ru} r \, du \\
&= \Pr(N_t \le k-1) && (3)
\end{aligned}$$

It then follows from (2) and (3) that

$$\Pr(N_t = k) = \Pr(N_t \le k) - \Pr(N_t \le k-1) = \frac{(rt)^k}{k!} e^{-rt}, \tag{4}$$

that is, N_t has the Poisson distribution with mean rt. This suggests, but does not prove, that $\{N_t, t \ge 0\}$ is a Poisson process. While this is the case, we shall not try to prove that this is so.

2 Poisson processes and random measures

Suppose we view our counting process as a way to assign an integer to an interval by the rule $[a,b) \to (N_b - N_a)$. We presume that

- For any integer m greater than 1, if $0 \le t_1 < t_2 \cdots < t_m$ the random variables $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots, N_{t_{m-1}} - N_{t_m}$ are mutually independent.

as in the definition of Poisson process, but we don't assume the Poisson distribution. Instead we assume that there is some $r > 0$ so that for $h > 0$

- $\Pr(N_{b+h} - N_b = 0) = 1 - rh + o(h)$;
- $\Pr(N_{b+h} - N_b = 1) = rh + o(h)$;
- $\Pr(N_{b+h} - N_b > 1) = o(h)$.

Here is how we can see that we must have the Poisson distribution for $N_b - N_a$. Choose a non-negative integer k and suppose that $n > k$. Put $h = (b-a)/n$ and $X_j = N_{a+jh} - N_{a+jh-h}$. These X_j are independent and $X_1 + \ldots + X_n = N_b - N_a$. What is more,

$$\{k \text{ out of } n \text{ of the } X_k \text{ equal } 1, \text{ the rest equal } 0\} \subset \{N_b - N_a = k\}$$

so

$$\Pr(N_b - N_a = k) \ge \binom{n}{k} \left(r\frac{b-a}{n} + o(1/n) \right)^k \left(1 - r\frac{b-a}{n} + o(1/n) \right)^{n-k}$$

Letting $n \to \infty$ we see that

$$\Pr(N_b - N_a = k) \ge \frac{r^k (b-a)^k}{k!} e^{-r(b-a)}$$

We have equality since if we did not, we would have the following contradiction:

$$1 = \sum_{k=0}^{\infty} \Pr(N_b - N_a = k) > \sum_{k=0}^{\infty} \frac{r^k(-a)^k}{k!} e^{r(b-a)} = 1$$

3 Poisson Processes and Differential Equations

Here is a third way to get at Poisson Processes. For each integer k put $p_k(t) = \Pr(N_t = k)$. We may proceed as follows, where we assume that $h > 0$, and we have the same assumptions as in the previous section.

$$\begin{aligned}
p_0(t+h) &= \Pr(N_{t+h} = 0) \\
&= \Pr(N_t = 0, N_{t+h} = 0) \\
&= \Pr(N_t = 0, N_{t+h} - N_t = 0) \\
&= \Pr(N_t = 0)\Pr(N_{t+h} - N_t = 0) \\
&= p_0(t)(1 - rh - o(h)).
\end{aligned}$$

By rearranging terms we get

$$\frac{p_0(t+h) - p_0(t)}{h} = -rp_0(t) + \frac{o(h)}{h}p_0(t)$$

If we let $h \to 0^+$ and assume for the moment that the existence of a righthand derivative of $p_0(t)$ makes $p_0(t)$ differentiable, we have

$$p_0'(t) = -rp_0(t).$$

Since $p_0(0) = \Pr(N_0 = 0) = 1$ we must have $p_0(t) = e^{-rt}$. Among other things, this shows that if $T_1 = \max\{t : N_t = 0\}$ then $\Pr(T_1 > t) = e^{-rt}$, that is, T_1 has an exponential distribution!

Now, let us look at $p_1(t)$. We will make the same assumption about differentiability as with p_0.

$$\begin{aligned}
p_1(t+h) &= \Pr(N_{t+h} = 1) \\
&= \Pr(N_t = 0, N_{t+h} = 1) + \Pr(N_t = 1, N_{t+h} = 1) \\
&= \Pr(N_t = 0, N_{t+h} - N_t = 1) + \Pr(N_t = 1, N_{t+h} - N_t = 0) \\
&= \Pr(N_t = 0)\Pr(N_{t+h} - N_t = 1) + \Pr(N_t = 1)\Pr(N_{t+h} - N_t = 0) \\
&= p_0(t)(rh + o(h)) + p_1(t)(1 - rh + o(h)).
\end{aligned}$$

By rearranging terms and using $p_0(t) = e^{-rt}$ we come to

$$\frac{p_1(t+h) - p_1(t)}{h} + rp_1(t) = re^{-rt} + \frac{o(h)}{h}p_0(t) + \frac{o(h)}{h}p_1(t).$$

Again, let $h \to 0^+$ obtain

$$p_1'(t) + tp_1(t) = re^{-rt}$$

while $p_1(0) = \Pr(N_0 = 1) = 0$. This differential equation may be solved by using the integrating factor e^{rt} (how convenient!) to get $p_1(t) = rte^{-rt}$.

Now for the general case. This will be a little trickier as it will involve the pinching theorem to get our differential equation. Suppose that k is a positive integer.

$$\begin{aligned}
p_k(t+h) &= \Pr(N_{t+h} = k) \\
&= \Pr(N_{t+h} - N_t = 0, N_{t+h} = k) + \Pr(N_{t+h} - N_t = 1, N_{t+h} = k)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \Pr(N_{t+h} - N_t \geq 2, N_{t+h} = k) \\
&= \Pr(N_t = k, N_{t+h} - N_t = 0) + \Pr(N_t = k-1, N_{t+h} - N_t = 1) \\
&\quad + \Pr(N_{t+h} - N_t \geq 2, N_{t+h} = k) \\
&= \Pr(N_t = k) \Pr(N_{t+h} - N_t = 0) + \Pr(N_t = k-1) \Pr(N_{t+h} - N_t = 1) \\
&\quad + \Pr(N_{t+h} - N_t \geq 2, N_{t+h} = k) \\
&= p_k(t)(1 - rh + o(h)) + p_{k-1}(t)(rh + o(h)) + \Pr(N_{t+h} - N_t \geq 2, N_{t+h} = k).
\end{aligned}
$$

Since

$$0 \leq \Pr(N_{t+h} - N_t \geq 2, N_{t+h} = k) \leq \Pr(N_{t+h} - N_t \geq 2) = o(h)$$

we have

$$p_k(t)(1-rh+o(h))+p_{k-1}(t)(rh+o(h)) \leq p_k(t+h) \leq p_k(t)(1-rh+o(h))+p_{k-1}(t)(rh+o(h))+o(h).$$

Rearranging terms we have

$$0 \leq \frac{p_k(t+h) - p_k(t)}{h} + rp_k(t) - rp_{k-1}(t) \leq \frac{o(h)}{h}\,(p_k(t) + p_{k-1}(t)) \leq \frac{o(h)}{h}$$

so by letting $h \to 0^+$ we get

$$p_k'(t) + rp_k(t) = rp_{k-1}(t)$$

with $p_k(0) = 0$. We can show recursively that

$$p_k(t) = \frac{(rt)^k}{k!} e^{-rt}.$$