Introduction to Martingales

1 Fair games

Let T be a subset of $[0, \infty)$, and suppose that $\{X_t, t \in T\}$ is a collection of real-valued random variables. Suppose that $f : T \times (-\infty, \infty) \to (-\infty, \infty)$ is continuous. Define $Y_t = f(t, X_t)$ and assume that $\mathrm{E}[|Y_t|] < \infty$ for each $t \in T$. We say that the Y_t form a martingale relative to the X_t if **martingale** if for any increasing sequence t_k of elements of T we have

$$\mathrm{E}[Y_{t_{n+1}} | X_{t_n} = x_n, X_{t_{n-1}} = x_{n-1}, \ldots, X_{t_0} = x_0] = f(t_n, x_n).$$

The idea we are trying to capture is that of a **fair game**. It seems reasonable to say that a game of chance Y_t whose randomness is driven by X_t is fair if given the history of the randomness to date, that is, given

$$\{X_{t_n} = x_n, X_{t_{n-1}} = x_{n-1}, \ldots, X_{t_0} = x_0\}$$

the expected value of the game at time t_{n+1} should be the value of the game at time t_n. Think, for example, of the game as poker, and the randomness is the random order of the deck of cards, or the game as craps, and the randomness is the rolls of the dice.

2 Some examples

Here are some examples.

2.1 Sums of independent random variables

Suppose that D_k are independent random variables, each with mean 0. Let T be the non-negative integers, $X_0 = 0$ and $X_n = D_1 + \cdots + D_n$. Then we take $f(t, x) = x$ and

$$\begin{aligned} E[X_{n+1} | X_n = x_n, \ldots, X_0 = 0] &= E[D_{n+1} + x_n | X_n = x_n, \ldots, X_0 = 0] \\ &= E[D_{n+1} | X_n = x_n, \ldots, X_0 = 0] + x_n \\ &= E[D_{n+1}] + x_n \\ &= x_n. \end{aligned}$$

2.2 Branching Processes

Suppose that X_n is a branching process where μ is the mean of the offspring distribution. Again, let T be the non-negative integers and let $Y_n = \mu^{-n} X_n$, that is, $f(t, x) = \mu^{-t} x$.

$$\begin{aligned} E[Y_{n+1} | X_n = x_n, \ldots, X_0 = 1] &= E[\mu^{-n-1} X_{n+1} | X_n = x_n, \ldots, X_0 = 1] \\ &= \mu^{-n-1} \mu x_n \\ &= \mu^{-n} x_n \end{aligned}$$

2.3 Discrete time, finite state Markov chains

Suppose that X_n is a finite state discrete time Markov chain with transition matrix P. Suppose that $P\vec{g} = a\vec{g}$ with $a \neq 0$, that is $\vec{g}$ is an eigenvector for P with eigenvalue a. Define $f(t, x) = a^{-t}\vec{g}_x$, where $\vec{g}_x$ is the x^{th} coordinate of $\vec{g}$. Then

$$\begin{aligned}
\mathrm{E}[Y_{n+1}|X_n = x, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] &= \mathrm{E}[Y_{n+1}|X_n = x] \\
&= \sum_s P_{x,s} a^{-n-1} \vec{g}_s \\
&= a^{-n-1} a \vec{g}_x \\
&= f(n, x)
\end{aligned}$$