Limiting and Stationary Probabilities

We want to address the question of when $\Pr(X_n = k|X_0 = i)$ converges when $n \to \infty$.

1 Ergodic Markov Chains

There is no chance that $(P^n)_{i,i}$ converges to a positive limit if i is not aperiodic. The same can be said if i is transient. We need some additional conditions for interesting convergence results.

For a given recurrent state i, define $T_i = \min\{n : X_n = i\}$. If $\mathrm{E}[T_i|X_0 = i] < \infty$ we say that the state i is **positive recurrent**. If a state i is both positive recurrent and aperiodic we say that the state is **ergodic**. It can be shown that the ergodic property is a class property (see *Basic Probability Theory* by Robert Ash).

We have the following theorem.

Theorem 1 (Ergodic Theorem) *If the Markov chain X_n with state space S is irreducible and all its states are ergodic then*

1. *There exist unique probability mass function π on S such that for each $s \in S$*

$$\pi_s = \sum_{x \in S} \pi_x P_{x,s},$$

or, in matrix form, $\vec{\pi} = \vec{\pi} P$.

2. *For each $x \in S$*

$$\lim_{n\to\infty} (P^n)_{x,s} = \pi_s.$$

In matrix form, there is a matrix Q with all rows equal to $\vec{\pi}$ and with $P^n \to Q$ as $n \to \infty$.

3. *For each $s \in S$ let $I_s(x) = 1$ if $x = s$ and 0 otherwise. Then*

$$\Pr\left(\lim_{n\to\infty} \frac{I_s(X_1) + \cdots + I_s(X_n)}{n} = \pi_s\right) = 1,$$

that is, π_s is the long-run average time the process spends in state s.

The hypotheses of this theorem are satisfied by all irreducible aperiodic finite state Markov chains.

One case where the π_s are easy to calculate is when the transpose of the transition matrix is also a transition matrix. If the transition matrix is finite dimensional and the chain is irreducible then $\pi_s = 1/N$ where N is the number of states. See, for example, problems 20, 21 and 22.

2 Stationary Distributions

We say that a probability mass function ν defined on the state space S is stationary if it the case that if $\Pr(X_0 = s)\nu_s$ then $\Pr(X_1 = s) = \nu(s)$ for all s in S. This is the same as saying the $\vec{\nu}P = \nu$. Hence, any limiting distribution $\vec{\pi}$ is a stationary distribution, but not every stationary distibution is a limiting distribution. The ergodic theorem gives conditions where the limiting distribution is the only stationary distribution.

3 The gambler's ruin problem

Consider the Markov chain on the the set $\{0, 1, \ldots, N\}$ where N is a least 2 and with the transition matrix $P_{0,0} = P_{N,N} = 1$, and for $s \neq 0, N$, $P_{s,s-1} + P_{s,s+1} = 1$. Let $p = P_{s,s+1} \in (0, 1)$. This Markov chain is called the gambler's ruin for the following reason. Let X_n denote the number of dollars a gambler has. Each time he plays he wins a dollar with probability p and loses a dollar with probability $1 - p$. If $X_n = 0$ or $X_n = N$ the game is over. The question is how to determine the probabilty that the state 0 is ever reached (it is absorbing) if the gambler starts at state s.

Denote this probability by f_s. It is clear that $f_0 = 1$ and $f_N = 0$. By considering one play of the game we see that $f_s = (1 - p)f_{s-1} + pf_{s+1}$ for $0 < s < N$. To solve for f_s, we guess a solution of the form $f_s = r^s$. Then $r^s = qr^{s-1} + pr^{s+1}$ where $q = 1 - p$. Dividing though by r^{s-1} we see that $r = q + pr^2$, so $r = 1$ or $r = q/p$. We consider first the case where $q \neq p$. Then $f_s = A + B(q/p)^s$. Since $f_0 = 1$ while $f_N = 0$ we can solve for A and B in terms of p, q and N to get

$$f_s = 1 - \frac{1 - (q/p)^s}{1 - (q/p)^N}. \tag{1}$$

To see what happens when $p = q$, let $h = q/p$ and let $h \to 1$ in (1). Since

$$\frac{1 - r^s}{1 - r^N} = \frac{1 + r + \cdots + r^{s-1}}{1 + r + \cdots + r^{N-1}},$$

we see that as h approaches 1, f_s approaches $(N - s)/N$ which we can show directly satisfies $f_s = (1/2)f_{s-1} + (1/2)f_{s+1}$ as well as the boundary conditions $f_0 = 1$ and $f_N = 0$.