Class Properties

Many important properties of the states of a Markov chain are shared by the members of the same communicating class. Such properties are called **class properties**.

1 Recurrence and transience are class properties

This is pretty easy to see. Suppose that i is recurrent, and j communicates with i. Then there are positive integers a and b such that $(P^a)_{i,j} > 0$ and $(P^b)_{j,i} > 0$. Therefore

$$\sum_{n=1}^{\infty}(P^n)_{j,j} \geq \sum_{n=1+a+b}^{\infty}(P^n)_{j,j} \geq \sum_{n=1}^{\infty}(P^b)_{j,i}(P^n)_{i,i}(P^a)_{i,j} = (P^b)_{j,i}(P^a)_{i,j}\sum_{n=1}^{\infty}(P^n)_{i,i} = \infty$$

Therefore j is recurrent too. If i is transient, then j cannot be recurrent, for it were, then since i communicates with j, i would be recurrent as well. Hence, in a finite state closed class of a Markov chain, all states are recurrent.

2 Periodicity

We observed in the simple random walk that if i and j are states and $|i-j|$ is an even number, then j can only be reached from i in an even number of states. In particular, we can only return to i from i in an even number of states. This is important if we are to study $\lim_{n\to\infty}\Pr(X_n = i|X_0 = i)$, for all the terms in the sequence $\Pr(X_n = i|X_0 = i)$ are zero when n is an odd number, so the only possible limit would be zero. While this is the case for this random walk, consider the random walk on the face of clock, with the states the hours, and the rule that at each step we either advance an hour with probability p or lose an hour with probability $1-p$. Again, after an odd number of steps the parity of the state is unchanged, but the limiting probability cannot be 0. This brings us to the notion of the period of a state.

For a given state i, let $D_i = \{n > 0 : \Pr(X_n = i|X_0 = i) > 0\}$. If D_i is not the empty set, we say that d_i is the period of i if d_i is the greatest common divisor of the elements of D_i. (If D_i is empty, we say that the period of i is 0.) In the two random walk models, each state has period 2. On the other hand, suppose that our clock face had an odd number of number on it, and $0 < p < 1$. Then starting from 1 we could reach 1 in an odd number of steps by going around the clock in the clockwise direction, which would require an odd number of steps, or by going from 1 to 2 and then 2 to 1, which is 2 steps. Since the greates common divisor of 2 and any odd number is 1 the period of 1 is 1. The same would be true of every state.

Period is also a class property.

Theorem 1 *If i and j communicate then $d_i = d_j$.*

To see why, since i and j communicate then neither D_i nor D_j is empty. Suppose $s \in D_i$. Then there exist a and b so that $b+s+a \in D_j$ and $a+2s+b \in D_j$ as well. Therefore d_j divides $a+s+b$ and $a+2s+b$, so d_j divides s. By the definition of greatest common divisor, d_j divides d_i. Now reverse the roles of i and j, so d_i divides d_j, showing that $d_i = d_j$.