Computing Probabilities Associated with Markov Chains

1 The basic example

Suppose that X_n is a Markov with transition matrix P and state space S. We suppose also that $\alpha_s^{(n)} := \Pr(X_n = s)$ for each $s \in S$.

We can make the following calculation:

$$\begin{aligned}
\alpha_s^{(0)} &= \Pr(X_n = s) \\
&= \sum_{a \in S} \Pr(X_n = s, X_{n-1} = a) \\
&= \sum_{a \in S} \Pr(X_n = s | X_{n-1} = a) \Pr(X_{n-1} = a) \\
&= \sum_{a \in S} \alpha_a^{(n-1)} P_{a,s}
\end{aligned}$$

which we can summarize in matrix form as

$$\alpha^{(n)} = \alpha^{(n-1)} P. \tag{1}$$

By iterating (1) we see that

$$\alpha^{(n)} = \alpha^{(0)} P^n. \tag{2}$$

Thus be able to analyse the behavior of powers of a fixed matrix is essential to the study of Markov chains.

2 Joint distributions for Markov chains

Suppose we want to know $\Pr(X_2 = c, X_1 = b, X_0 = a)$ and we know the transition matrix of the Markov chain, P, and its initial distribution, $\alpha^{(0)}$. We reason as follows.

$$\begin{aligned}
\Pr(X_2 = c, X_1 = b, X_0 = a) &= \Pr(X_2 = c | X_1 = b, X_0 = a) \Pr(X_1 = b, X_0 = a) \\
&= \Pr(X_2 = c | X_1 = b) \Pr(X_1 = b, X_0 = a) \quad \text{Markov chain!!!} \\
&= \Pr(X_2 = c | X_1 = b) \Pr(X_1 = b | X_0 = a) \Pr(X_0 = a) \\
&= \alpha_a^{(0)} P_{a,b} P_{b,c}
\end{aligned}$$

3 The Chapman-Kolmogorov equations

The following example conveys the essential idea:

$$\begin{aligned}
\Pr(X_{n+2} = c | X_n = a) &= \frac{\Pr(X_{n+2} = c, X_n = a)}{\Pr(X_n = a)} \\
&= \sum_{b \in S} \frac{\Pr(X_{n+2} = c, X_{n+1} = b, X_n = a)}{\Pr(X_n = a)} \\
&= \sum_{b \in S} \frac{\Pr(X_{n+2} = c | X_{n+1} = b, X_n = a) \Pr(X_{n+1} = b, X_n = a)}{\Pr(X_n = a)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{b \in S} \Pr(X_{n+2} = c | X_{n+1} = b) \Pr(X_{n+1} = b | X_n = a) \\
&= \sum_{b \in S} P_{a,b} P_{b,c} \\
&= \left(P^2\right)_{a,c}.
\end{aligned}
$$

By similar calculations we can show that for any non-negative integers k and n and any pair of states a and c,

$$\Pr(X_{n+k} = c | X_n = a) = \left(P^k\right)_{a,c}.$$