Homework 10: Brownian Motion and the Black Scholes Formula

Through these problems B_t denotes the standard Brownian motion, that is, $\mathrm{E}[B_t] = 0$ and $\mathrm{Var}[B_t] = t$.

One point problems

1. Verify that for each of the following functions $f(t,x)$ that $\mathrm{E}[f(t+h, B_{t+h})|B_t = b] = f(t,b)$:
   - $f(t,x) = x^2 - t$
   - $\exp\left(-\frac{t}{2} + x\right)$
2. Verify that for each of the following functions $f(t,x)$ that $\mathrm{E}[f(t+h, B_{t+h})|B_t = b] = f(t,b)$:
   - $\exp\left(\frac{t}{2}\right)\cos(x)$
   - $x^3 - 3tx$.
3. Suppose that for $t > 0$ that
$$p(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$
   Note that p is the density for B_t. Show that
$$\frac{\partial p}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(t,x).$$
   This partial differential equation is called the **Heat Equation**.
4. Suppose that $g : (-\infty, \infty) \to (-\infty, \infty)$ is a continuous function and that there is some constant K so that $|g(x)| \leq K$ for all real numbers x. Define $f(t,x) = \mathrm{E}[g(x + B(t))]$. Show that
$$f(t,x) = \int_{-\infty}^{\infty} g(x+y)p(t,y)\,dy = \int_{-\infty}^{\infty} g(z)p(t,x-z)\,dz.$$
   Assuming that differentiating under the integral sign is permissable, show that $\mathrm{E}[g(x + B_t)]$ is also a solution to the heat equation.
5. Determine $\Pr(B_{t+s} > B_t)$ if $s > 0$ and $t > 0$.
6. Suppose that $t \in [0,1]$ and define $X_t = B_t - tB_1$. X_t is called the Brownian Bridge because $X_0 = X_1 = B_0 = 0$. If $t \in (0,1)$ show that X_t has a normal distribution, and compute its mean and variance. Hint: Show that $X_t = (1-t)B_t + t(B_t - B_1)$.
7. Suppose that $0 < s < t$. Compute the covariance of B_t and B_s.
8. Suppose that you wanted to buy a European call option on IBM with a strike price of \$100 with expiration in 3 months. If the current price is \$127.50 and the guarenteed interest rate is 2% per year, what is the Black Scholes price of this option if the annual volatility is 0.01?

9. If you go to http://www.maths.ox.ac.uk/ howison/o10/excel/IBM.xls you will find option prices for IBM for 1996/1997. Assuming that the April prices are for 6 month expiration dates and the that the annual guaranteed interest rate was 2% for that six month window, what was the range of implied volatilities for the listed exercise prices?

10. Use a graphing utility to graph the Black-Scholes price of an option to buy a 3 month European call on IBM versus the implied annual volatility if the current price is \$100.00 per share, the strike price is \$110 dollars per share, and the guarenteed annual interest is 4.5%. Draw second graph if the strike price is changed to \$90.00 per share.

11. Set up an Excel spread sheet to implement Black-Scholes pricing. Use your sheet to price some options from some current data on IBM or some other company. For example, you can get data at www.finance.yahoo.com

Two point problems

1. If $0 < s < t < 1$ and X is the Brownian bridge, find the covariance of X_s and X_t.

2. Suppose that on the interval $[0, T]$ that U and V are non-negative functions and that U/V is strictly increasing. Define Y_t by $Y_t = V(t)B_{U(t)/V(t)}$. Show that the covariance of Y_t and Y_s is $U(s)V(t)$ if $0 < s < t \leq T$.

3. Find functions $U(t)$ and $V(t)$ so that $V(t)B_{U(t)/V(t)}$ has the same distribution as the Brownian bridge X_t in problem 5 above and $U(t)/V(t)$ is increasing for $0 < t < 1$.

4. Find functions U and V so that $Y_t := V(t)B_{U(t)/V(t)}$ has a constant variance and $U(t)/V(t)$ is increasing for $-\infty < t < \infty$.

5. Suppose that

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^f}{\partial x^2} = 0$$

for $t > 0$ and $-\infty < x < \infty$. Examples of such functions are given in the first one point problem. Assuming that one can interchange integration and differentiation and that all integrals are convergent, show that

$$\frac{\partial}{\partial t}\mathrm{E}[f(t, B_t)] = 0$$