Homework 9: Review of Poisson Processes and Continuous Time Markov Chains

One point problems

1. Suppose that N_t is a Poisson process with rate 5. Compute $\Pr(N_3 = 8, N_7 = 12)$.

   We need to use the independent, stationary increments property:

$$\begin{aligned}
\Pr(N_3 = 8, N_7 = 12) &= \Pr(N_3 = 8, N_7 - N_3 = 4) \\
&= \Pr(N_3 = 8)\Pr(N_7 - N_3 = 4) \\
&= \frac{15^8 \exp(-15)}{8!}\frac{20^4 \exp(-20)}{4!} \\
&= \frac{15^8 20^4}{8!4!}\exp(-35)
\end{aligned}$$

2. Suppose that N_t is a Poisson process with rate 7. Compute $\Pr(N_3 = 8 | N_{10} = 22)$.

   We need to use the independent, stationary increments property and the definition of conditional probability:

$$\Pr(N_3 = 8 | N_{10} = 22) = \frac{\Pr(N_3 = 8, N_{10} = 22)}{\Pr(N_{10} = 22)}.$$

   As in the previous problem:

$$\begin{aligned}
\Pr(N_{10} = 22) &= \frac{70^{22}}{22!}\exp(-70) \\
\Pr(N_3 = 8, N_{10} = 22) &= \frac{21^8 49^{14}}{8!14!}\exp(-70)
\end{aligned}$$

   so

$$\Pr(N_3 = 8 | N_{10} = 22) = \begin{pmatrix} 22 \\ 8 \end{pmatrix}\left(\frac{3}{10}\right)^8\left(\frac{7}{10}\right)^{14}$$

3. Suppose that N_t is a Poisson process with rate 2, and that X_k are independent, identically distributed random variables with $\Pr(X_k = 0) = p > 0$ and $\Pr(X_k = 1) = 1 - p > 0$. Let

$$Y_t = \sum_{k=1}^{N_t} X_k.$$

   Find $\Pr(N_6 = 5 | Y_6 = 3)$.

   We use the property that random classification of Poisson events gives independent Poisson processes.

   Let

$$Z_t = \sum_{k=1}^{N_t}(1 - X_k).$$

Both Y_t and Z_t are Poisson processes, and they are independent of one another. Furthermore, $N_t = X_t + Y_t$. Therefore,

$$\begin{aligned}
\Pr(N_6 = 5|Y_6 = 3) &= \Pr(Y_6 + Z_6 = 5|Y_6 = 3) \\
&= \frac{\Pr(Y_6 + Z_6 = 5, Y_6 = 3)}{\Pr(Y_6 = 3)} \\
&= \frac{\Pr(Z_6 = 2, Y_6 = 3)}{\Pr(Y_6 = 3)} \\
&= \frac{\Pr(Z_6 = 2)\Pr(Y_6 = 3)}{\Pr(Y_6 = 3)} \\
&= \Pr(Z_6 = 2) \\
&= \frac{(2p))^2}{2!}\exp(-2p))
\end{aligned}$$

4. The arrival of claims at an insurance company follows a Poisson process. On average the company gets 100 claims per week. The claims follow an exponential distribution with mean \$700.00. They offer two types of policies. The first type has no deductible and the second has a \$250.00 deductible. If the claim sizes and policy types are independent of each other and of the number of claims, and twice as many policy holders have deductibles as not, what is the mean and variance of the liability of the company in any 13 week period?

   This is an extension of the previous problem. There are two separate Poisson processes at work here. The first is the arrival of the claims on the no deductible policies, which arrive at a rate of 100/3 per week, and the second is the arrival of claims on the deductible policies, which arrive at a rate of 200/3 per week. Denote this first Poisson process by N_t (for no deductible) and the second by D_t (for deductible). These two processes are independent.

   Let X_k be the company liability on a claim under a deductible policy, and let Y_k be the company liability on a claim under a no-deductible policy. Y_k is exponential, with mean 700. $X_k = \max(0, X'_k - 250)$, where X'_k is exponential with mean 700.

$$\begin{aligned}
\mathrm{E}[X_1] &= \int_0^\infty \max(0, x - 250)\exp(-x/700)\frac{1}{700}dx \\
&= \int_{250}^\infty x - 250\exp(-x/700)\frac{1}{700}dx \\
&= \exp(-250/700)\int_0^\infty y\exp(-y/700)\frac{1}{700}dy \\
&= 700\exp(-5/14) \\
\mathrm{E}[X_1^2] &= \int_0^\infty (\max(0, x - 250))^2\exp(-x/700)\frac{1}{700}dx \\
&= \int_{250}^\infty (x - 250)^2\exp(-x/700)\frac{1}{700}dx \\
&= \exp(-250/700)\int_0^\infty y^2\exp(-y/700)\frac{1}{700}dy \\
&= 980,000\exp(-5/14) \\
\mathrm{Var}(X_k) &= 490,000(2\exp(-5/14) - \exp(-5/7)),
\end{aligned}$$

while $\mathrm{E}[Y_1] = 700$ and $\mathrm{Var}[Y_1] = 490,000$.

$$\begin{aligned} L_D(t) &= \sum_{k=1}^{D_t} X_k \\ L_N(t) &= \sum_{k=1}^{N_t} Y_k \\ L(t) &= L_D(t) + L_N(t) \end{aligned}$$

We need the mean and variance of $L(t)$, and $L(t)$ is the sum of two independent random variables, so $\mathrm{E}[L(t)] = \mathrm{E}[L_D(t)] + \mathrm{E}[L_N(t)]$ and $\mathrm{Var}[L(t)] = \mathrm{Var}[L_D(t)] + \mathrm{Var}[L_N(t)]$. We have

$$\begin{aligned} \mathrm{E}[L_D(t)] &= \mathrm{E}[D_t]\mathrm{E}[X_1] \\ \mathrm{E}[L_N(t)] &= \mathrm{E}[N_t]\mathrm{E}[Y_1] \\ \mathrm{Var}[L_D(t)] &= \mathrm{E}[D_t]\mathrm{Var}[X_1] + \mathrm{E}[X_2]^2\mathrm{Var}[D_t] \\ \mathrm{Var}[N_D(t)] &= \mathrm{E}[N_t]\mathrm{Var}[Y_1] + \mathrm{E}[Y_1]^2\mathrm{Var}[L_t] \end{aligned}$$

5. It takes Fred an average of 6 hours per week to complete his Math 571 homework, and it takes Ethel an average of 5 hours per week to complete her Math 362 homework. Assuming that the time to completion for each student is exponentially distributed, if Fred starts an hour before Ethel, what is the probability that he finishes before she does?

   The issue is whether or not Fred finishes before Ethel starts. Therefore the probability that Fred finishes before Ethel is

$$(1 - \exp(-1/6)) + \exp(-1/6)\frac{5}{11}.$$

6. A random variable W is said to have a Weibull distribution if there is some $p > 0$ so that W^p has an exponential distribution. Find the formula for the survival function and hazard function for a Weibull random variable in terms of this exponent p and the mean μ of the resulting exponential random variable W^p. (In other words, W^p is exponential and $\mathrm{E}[W^p] = \mu$.)

   Suppose that such a p exists. Then

$$\begin{aligned} \Pr(W > x) &= \Pr(W^p > x^p) \\ &= \exp(-x^p/\mu) \end{aligned}$$

   The hazard function of a random variable is its density divided by its survival function. This is the same as the negative of the derivative of the log of the survival function. Hence the survival function of W is px^{p-1}/μ (for $x > 0$).

7. Suppose that N_t is a Poisson process with rate 1 and $M_t = N_{m(t)}$ is the inhomogeneous Poisson process with mean function $m(t) = 3t + \sin(t)$. Find $\Pr(M_\pi = 7, M_{2\pi} = 12)$.

   Here we use the independent increments property in the same way as in the first problem.

$$\begin{aligned} \Pr(M_\pi = 7, M_{2\pi} = 12) &= \Pr(M_\pi = 7, M_{2\pi} - M_\pi = 5) \\ &= \Pr(M_\pi = 7)\Pr(M_{2\pi} - M_\pi = 5) \end{aligned}$$

$$\begin{aligned}
&= \Pr(N_{3\pi} = 7)\Pr(N_{6\pi} - N_{3\pi} = 5) \\
&= \frac{(3\pi)^7}{7!}\exp(-3\pi)\frac{(3\pi)^5}{5!}\exp(-3\pi) \\
&= \frac{(3\pi)^{12}}{7!5!}\exp(-6\pi)
\end{aligned}$$

8. Consider a Birth and Death process with $\lambda(s) = s+1$ and $\delta(s) = 2s$. Find the limiting distribution for this process.

   We know that

$$\begin{aligned}
\pi_s &= \pi_0 \prod_{k=1}^{s} \frac{\lambda(k-1)}{\delta(k)} \\
&= \pi_0 \prod_{k=1}^{s} \frac{k}{2k} \\
&= \left(\frac{1}{2}\right)^s
\end{aligned}$$

   Therefore

$$1 = \sum_{s=0}^{\infty} \pi_s = \pi_0 \left(1 + \sum_{s=1}^{\infty} \left(\frac{1}{2}\right)^s\right) = 2\pi_0.$$

   Therefore $\pi_s = (1/2)^{s+1}$, the limiting distribution is geometric!

9. Consider a Birth and Death process X_t on $\{0,1,2,3\}$ with $\lambda(s) = (3-s)^2$ and $\delta(s) = s^2 + s$. Find $\mathrm{E}[X_t]$ and $\mathrm{Var}[X_t]$ assuming that $\Pr(X_0 = 3) = 1$.

   The rate matrix for this process is

$$\Lambda = \begin{bmatrix} -9 & 9 & 0 & 0 \\ 2 & -6 & 4 & 0 \\ 0 & 6 & -7 & 1 \\ 0 & 0 & 12 & -12 \end{bmatrix}$$

   We have

$$\begin{bmatrix} -9 & 9 & 0 & 0 \\ 2 & -6 & 4 & 0 \\ 0 & 6 & -7 & 1 \\ 0 & 0 & 12 & -12 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -5 \\ -12 \end{bmatrix} = 9\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 7\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

   Therefore

$$\frac{d}{dt}\mathrm{E}[X_t] = 9 - 7\mathrm{E}[X_t].$$

   We know that $\mathrm{E}[X_0] = 3$. We use integrating factors to solve this differential equation.

$$\begin{aligned}
\frac{d}{dt}\mathrm{E}[X_t] &= 9 - 7\mathrm{E}[X_t] \\
\frac{d}{dt}\mathrm{E}[X_t] + 7\mathrm{E}[X_t] &= 9 \\
\exp(7t)\left(\frac{d}{dt}\mathrm{E}[X_t] + 7\mathrm{E}[X_t]\right) &= 9\exp(7t)
\end{aligned}$$

$$
\begin{aligned}
\exp(7t)\frac{d}{dt}\mathrm{E}[X_t] + 7\exp(7t)\mathrm{E}[X_t]) &= 9\exp(7t) \\
\frac{d}{dt}\left(\exp(7t)\mathrm{E}[X_t]\right) &= 9\exp(7t) \\
\exp(7t)\mathrm{E}[X_t] - \exp(0)\mathrm{E}[X_0] &= \frac{9}{7}\left(\exp(7t) - 1\right) \\
\exp(7t)\mathrm{E}[X_t] - 3 &= \frac{9}{7}\left(\exp(7t) - 1\right) \\
\mathrm{E}[X_t] &= \frac{9}{7} + \frac{12}{7}\exp(-7t)
\end{aligned}
$$

To find the variance, we will find $\mathrm{E}[X_t^2]$. This time we use a second approach. Put $M_2(t) = \mathrm{E}[X_t^2]$.

$$
\begin{aligned}
\mathrm{E}[X_{t+h}^2|X_t = s] &= (s+1)^2\lambda(s)h + s^2(1 - \lambda(s)h - \delta(s)h) + (s-1)^2\delta(s)h + o(h) \\
&= s^2 + 2s(\lambda(s) - \delta(s))h + (\lambda(s) + \delta(s))h + o(h) \\
&= s^2 + 2s(9 - 7s)h + (9 - 5s + 2s^2)h + o(h) \\
&= s^2 + (-12s^2 + 13s + 9)h + o(h)
\end{aligned}
$$

so, since $M_2(0) = 9$,

$$
\begin{aligned}
M_2(t+h) &= M_2(t) + (-12M_2(t) + 13\mathrm{E}[X_t] + 9)h + o(h) \\
\frac{M_2(t+h) - M_2(t)}{h} &= -12M_2(t) + 13\mathrm{E}[X_t] + 9 + \frac{o(h)}{h} \\
M_2'(t) &= -12M_2(t) + 13\mathrm{E}[X_t] + 9 \\
M_2'(t) + 12M_2(t) &= 13\mathrm{E}[X_t] + 9 \\
&= \frac{180}{7} + \frac{156}{7}\exp(-7t) \\
(\exp(12t)M_2(t))' &= \frac{180}{7}\exp(12t) + \frac{156}{7}\exp(5t) \\
\exp(12t)M_2(t) - 9 &= \frac{15}{7}(\exp(12t) - 1) + \frac{156}{35}(\exp(5t) - 1) \\
M_2(t) &= 9\exp(-12t) + \frac{15}{7}(1 - \exp(-12t)) + \frac{156}{35}(\exp(-7t) - \exp(-12t))
\end{aligned}
$$

and

$$
\mathrm{Var}[X_t] = M_2(t) - (\mathrm{E}[X_t])^2.
$$

10. Suppose that the rate matrix for a continuous time Markov chain is

$$
\begin{bmatrix} -4 & 2 & 2 \\ 3 & -4 & 1 \\ 1 & 3 & -4 \end{bmatrix}.
$$

Find the limiting distribution for this Markov chain.

If we let Λ denote the rate matrix we want to solve $\vec{\pi}\Lambda = \vec{0}$ where the components of $\vec{\pi}$ sum to 1. This yields

$$
\vec{\pi} = \left[\frac{13}{37}, \frac{14}{37}, \frac{10}{37}\right].
$$

Two point problems

1. Suppose that $0 < p < (e-1)/e$, that N_t is a Poisson process with rate 1 and X_k are iid random variables, independent of the Poisson process with

$$\Pr(X_k = j) = \begin{cases} 1 + \log(1-p) & \text{if} \quad j = 0 \\ \dfrac{p^j}{j} & \text{if} \quad j = 1, 2, \ldots \end{cases}$$

Show that

$$Y_t = \sum_{k=1}^{N_t} X_k$$

has a negative binomial distribution. Hint: Use probability generating functions.

Let $P(s,t)$ be the probabilility generating function of N_t:

$$P(s,t) = \mathrm{E}[s^{N_t}] = \sum_{k=0}^{\infty} s^k \frac{t^k}{k!} \exp(-t) = \exp(t(s-1)).$$

Let $Q(u)$ be the probability generating function of X_k:

$$Q(u) = \mathrm{E}[u^{X_k}] = (1 + \log(1-p)) + \sum_{k=1}^{\infty} u^k \frac{p^k}{k} = 1 + \log(1-p) - \log(1-pu)$$

where $|pu| < 1$.

Now we get the probability generating function for Y_t:

$$\begin{aligned} \mathrm{E}[u^{Y_t}] &= \sum_{k=0}^{\infty} \mathrm{E}[u^{Y_t} I_{N_t=k}] \\ &= \Pr(N_t = 0) + \sum_{k=1}^{\infty} \mathrm{E}[u^{X_1+\cdots+X_k}] \Pr(N_t = k) \\ &= \sum_{k=0}^{\infty} Q(u)^k \Pr(N_t = k) \\ &= P(Q(u), t) \\ &= \exp(t(\log(1-p) - \log(1-pu))) \\ &= \left(\frac{1-p}{1-pu} \right)^t \end{aligned}$$

which is the probability generating function of the negative binomial distribution, that is,

$$\Pr(Y_t = k) = (1-p)^t \binom{-t}{k} (-p)^k.$$

See Asssignment 5 (http://www.uwm.edu/ ericskey/571material/add05.1.pdf) for details on the negative binomial distribution.

2. Suppose that X_t is a birth and death process on the finite state space $\{0, 1, \ldots, N\}$ with $\lambda(s) - \delta(s) = a + bs$ and $\lambda(s) + \delta(s) = a + cs + ds^2$. If $\mathrm{E}[X_0] = \mu_1$ and $\mathrm{E}[X_0^2] = \mu_2$, express $\mathrm{E}[X_t]$ and $\mathrm{E}[X_t^2]$ in terms of N, t, a, b, c, d, μ_1 and μ_2.

Note first that $\lambda(0) > 0$ and $\delta(N) > 0$, while $\delta(0) = \lambda(N) = 0$, and

$$\begin{aligned}\lambda(0) - \delta(0) &= a \\ \lambda(N) - \delta(N) &= a + bN\end{aligned}$$

so $b = -(\lambda(0) + \delta(N))/N < 0$.
Let $M_1(t) = \mathrm{E}[X_t]$. We have

$$\begin{aligned}\mathrm{E}[X_{t+h}|X_t = s] &= (s+1)\lambda(s)h + s(1 - \lambda(s)h - \delta(s)h) + (s-1)\delta(s)h + o(h) \\ &= s + (\lambda(s) - \delta(s))h + o(h) \\ &= s + (a + bs)h + o(h).\end{aligned}$$

Therefore

$$\begin{aligned}M_1(t+h) &= M_1(t) + (a + bM_1(t))h + o(h) \\ \frac{M_1(t+h) - M_1(t)}{h} &= a + bM_1(t) + \frac{o(h)}{h} \\ M_1'(t) &= a + bM_1(t)\end{aligned}$$

Since $\mathrm{E}[X_0] = \mu$ we have

$$M_1(t) = -\frac{a}{b} + \left(\mu + \frac{a}{b}\right)\exp(bt).$$

Now let $M_2(t) = \mathrm{E}[X_t^2]$.

$$\begin{aligned}\mathrm{E}[X_{t+h}|X_t = s] &= (s+1)^2\lambda(s)h + s^2(1 - \lambda(s)h - \delta(s)h) + (s-1)^2\delta(s)h + o(h) \\ &= s^2 + 2s(\lambda(s) - \delta(s))h + (\lambda(s) + \delta(s))h + o(h) \\ &= s^2 + (a + (2a+c)s + (2b+d)s^2)h + o(h).\end{aligned}$$

Therefore

$$\begin{aligned}M_2(t+h) &= M_2(t) + (a + (2a+c)M_1(t) + (2b+d)M_2(t))h + o(h) \\ \frac{M_2(t+h) - M_2(t)}{h} &= a + (2a+c)M_1(t) + (2b+d)M_2(t) + \frac{o(h)}{h} \\ M_2'(t) &= a + (2a+c)M_1(t) + (2b+d)M_2(t) \\ M_2'(t) - (2b+d)M_2(t) &= a + (2a+c)M_1(t) \\ &= a\frac{b-2a-c}{b} + (2a+c)\left(\mu + \frac{a}{b}\right)\exp(bt)\end{aligned}$$

The integrating factor for this equation is $\exp(-(2b+d)t)$:

$$\begin{aligned}(\exp(-(2b+d)t)M_2(t))' &= a\frac{b-2a-c}{b}\exp(-(2b+d)t) \\ &\quad +(2a+c)\left(\mu + \frac{a}{b}\right)\exp(-(b+d)t) \\ M_2(t)\exp(-(2b+d)t) - (\sigma^2 + \mu^2) &= \frac{a(b+c)}{2a+d}(\exp(-(2a+d)t) - 1) \\ &\quad +\frac{2b+d}{b-2a-d}\left(\mu + \frac{a}{b}\right)(\exp((b-2a-d)t) - 1)\end{aligned}$$

Therefore,

$$\begin{aligned} M_2(t) &= (\sigma^2+\mu^2)\exp((2a+d)t) + \frac{a(b+c)}{2a+d}(1-\exp((2a+d)t)) \\ &+ \frac{1}{b-2a-d}(2b+d)\left(\mu+\frac{a}{b}\right)(\exp(bt)-\exp((2a+d)t)-1) \end{aligned}$$

3. Suppose that X_t is a birth and death process on $\{0,1,2\}$ with $\lambda(s) = 4-s^2$ and $\delta(s) = s$. Find the transition matrix P_t of X_t.

   The rate matrix for this process is

$$\Lambda = \begin{bmatrix} -4 & 4 & 0 \\ 1 & -4 & 3 \\ 0 & 2 & -2 \end{bmatrix}$$

   The transition matrix is $\exp(t\Lambda)$. To compute this we observe that the characteristic polynomial of Λ is $p(t) = t(t^2+10t+22)$, so that Λ has eigenvalues of $0, -5+\sqrt{3}, -5-\sqrt{3}$. We have that

$$\begin{aligned} \Lambda &= \begin{bmatrix} 1 & 24 & 24 \\ 1 & -6+6\sqrt{3} & -6-6\sqrt{3} \\ 1 & -4\sqrt{3} & 4\sqrt{3} \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5+\sqrt{3} & 0 \\ 0 & 0 & -5-sqrt3 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 24 & 24 \\ 1 & -6+6\sqrt{3} & -6-6\sqrt{3} \\ 1 & -4\sqrt{3} & 4\sqrt{3} \end{bmatrix} \end{aligned}$$

   Therefore

$$\begin{aligned} &\exp(t\Lambda) \\ = &\begin{bmatrix} 1 & 24 & 24 \\ 1 & -6+6\sqrt{3} & -6-6\sqrt{3} \\ 1 & -4\sqrt{3} & 4\sqrt{3} \end{bmatrix}^{-1} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \exp((-5+\sqrt{3})t) & 0 \\ 0 & 0 & \exp((-5-\sqrt{3})t) \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 24 & 24 \\ 1 & -6+6\sqrt{3} & -6-6\sqrt{3} \\ 1 & -4\sqrt{3} & 4\sqrt{3} \end{bmatrix} \end{aligned}$$

4. Suppose that X_t is a continuous time Markov chain on $\{1,2,\ldots,N\}$ with rate matrix Λ. Suppose that $\vec{v} \in R^N$ and a are an eigenvector/eigenvalue pair for Λ, that is, $\Lambda\vec{v} = a\vec{v}$. Let $v(s)$ be the s^{th} coordinate of $\vec{v}$ and define $V_t = v(X_t)$. Find $\mathrm{E}[V_{t+h}|X_t = a]$.

5. A store owner is trying to decide between two ways of checking out customers. The first possibility is that he can invest in a high-tech cash register that will allow a single cashier to serve 20 customers per hour. The second possibility is that he can hire two cashiers to serve customers, each cashier working at the rate of 10 customers per hour. If customers appear to be checked out at the rate of 15 per hour, compare the two methods of serving the customers by computing the mean and variance of the number of customers waiting to be checked out.

In the first scenario, let s be the number of people in the system. Then $\delta(s) = 20$ for $s \geq 1$ and $\lambda(s) = 15$ for $s \geq 0$. Therefore

$$\pi_s = \left(\frac{15}{20}\right)^s \pi_0 = \left(\frac{3}{4}\right)^s \pi_0$$

so

$$1 = \pi_0 \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = 4\pi_0.$$

Therefore

$$\pi_s = \frac{1}{4}\left(\frac{3}{4}\right)^s.$$

Now, if the state is s then the number of people in line is $s-1$ if $s > 0$ and is 0 if $s = 0$. Therefore the expected number of people in line under this arrangement is

$$\sum_{s=1}^{\infty}(s-1)\pi_s = \sum_{s=1}^{\infty}(s-1)\frac{1}{4}\left(\frac{3}{4}\right)^s = \frac{9}{4}.$$

The second moment is

$$\sum_{s=1}^{\infty}(s-1)^2\pi_s = \sum_{s=1}^{\infty}(s-1)^2\frac{1}{4}\left(\frac{3}{4}\right)^s = \frac{63}{4}$$

so the variance is 171/16. That gives an approximate standard deviation of 3.269.

In the second scenario we still have $\lambda(s) = 15$ for $s \geq 0$, but now we have $\delta(1) = 10$ and $\delta(s) = 20$ for $s \geq 2$. Now

$$\begin{aligned}
\pi_1 &= \frac{15}{10}\pi_0 \\
&= \frac{3}{2}\pi_0 \\
\pi_s &= \frac{15}{20}\pi_{s-1} \\
&= \frac{3}{4}\pi_{s-1} \\
&= 2\left(\frac{3}{4}\right)^s \pi_0
\end{aligned}$$

for $s \geq 1$. Therefore

$$1 = \pi_0\left(1 + 2\sum_{s=1}\left(\frac{3}{4}\right)^s\right) = 7\pi_0$$

Therefore

$$\begin{aligned}
\pi_0 &= \frac{1}{7} \\
\pi_s &= \frac{2}{7}\left(\frac{3}{4}\right)^s
\end{aligned}$$

if $s \geq 1$. Now, the number of people in the queue is $s-2$ if $s \geq 2$ and is 0 otherwise. Therefore the expected number of the people in the queue is

$$\sum_{s=2}^{\infty}(s-2)\pi_s = \sum_{s=2}^{\infty}(s-2)\frac{2}{7}\left(\frac{3}{4}\right)^s = \frac{27}{14}.$$

while the second moment is

$$\sum_{s=2}^{\infty}(s-2)^2\pi_s = \sum_{s=2}^{\infty}(s-2)^2\frac{2}{7}\left(\frac{3}{4}\right)^s = \frac{27}{2}.$$

Therefore the variance in this case is 1976/196. That gives an approximate standard deviation of 3.127.

In comparison, the second option has both a lower expected number of people in line and a smaller standard deviation too.