

# Homework 9: Review of Poisson Processes and Continuous Time Markov Chains

## One point problems

1. Suppose that  $N_t$  is a Poisson process with rate 5. Compute  $\Pr(N_3 = 8, N_7 = 12)$ .
2. Suppose that  $N_t$  is a Poisson process with rate 7. Compute  $\Pr(N_3 = 8 | N_{10} = 22)$ .
3. Suppose that  $N_t$  is a Poisson process with rate 2, and that  $X_k$  are independent, identically distributed random variables with  $\Pr(X_k = 0) = p > 0$  and  $\Pr(X_k = 1) = 1 - p > 0$ . Let

$$Y_t = \sum_{k=1}^{N_t} X_k.$$

Find  $\Pr(N_6 = 5 | Y_6 = 3)$ .

4. The arrival of claims at an insurance company follows a Poisson process. On average the company gets 100 claims per week. The claims follow an exponential distribution with mean \$700.00. They offer two types of policies. The first type has no deductible and the second has a \$250.00 deductible. If the claim sizes and policy types are independent of each other and of the number of claims, and twice as many policy holders have deductibles as not, what is the mean and variance of the liability of the company in any 13 week period?
5. It takes Fred an average of 6 hours per week to complete his Math 571 homework, and it takes Ethel an average of 5 hours per week to complete her Math 362 homework. Assuming that the time to completion for each student is exponentially distributed, if Fred starts an hour before Ethel, what is the probability that he finishes before she does?
6. A random variable  $W$  is said to have a Weibull distribution if there is some  $p > 0$  so that  $W^p$  has an exponential distribution. Find the formula for the survival function and hazard function for a Weibull random variable in terms of this exponent  $p$  and the mean  $\mu$  of the resulting exponential random variable  $W^p$ . (In other words,  $W^p$  is exponential and  $E[W^p] = \mu$ .)
7. Suppose that  $N_t$  is a Poisson process with rate 1 and  $M_t = N_{m(t)}$  is the inhomogeneous Poisson process with mean function  $m(t) = 3t + \sin(t)$ . Find  $\Pr(M_\pi = 7, M_{2\pi} = 12)$ .
8. Consider a Birth and Death process with  $\lambda(s) = s + 1$  and  $\delta(s) = 2s$ . Find the limiting distribution for this process.
9. Consider a Birth and Death process  $X_t$  on  $\{0, 1, 2, 3\}$  with  $\lambda(s) = (3 - s)^2$  and  $\delta(s) = s^2 + s$ . Find  $E[X_t]$  and  $\text{Var}[X_t]$  assuming that  $\Pr(X_0 = 3) = 1$ .
10. Suppose that the rate matrix for a continuous time Markov chain is

$$\begin{bmatrix} -4 & 2 & 2 \\ 3 & -4 & 1 \\ 1 & 3 & -4 \end{bmatrix}.$$

Find the limiting distribution for this Markov chain.

## Two point problems

1. Suppose that  $0 < p < (e - 1)/e$ , that  $N_t$  is a Poisson process with rate 1 and  $X_k$  are iid random variables, independent of the Poisson process with

$$\Pr(X_k = j) = \begin{cases} 1 + \log(1 - p) & \text{if } j = 0 \\ \frac{p^j}{j} & \text{if } j = 1, 2, \dots \end{cases}$$

Show that

$$Y_t = \sum_{k=1}^{N_t} X_k$$

has a negative binomial distribution. Hint: Use probability generating functions.

2. Suppose that  $X_t$  is a birth and death process on the finite state space  $\{0, 1, \dots, N\}$  with  $\lambda(s) - \delta(s) = a + bs$  and  $\lambda(s) + \delta(s) = a + cs + ds^2$ . If  $E[X_0] = \mu_1$  and  $E[X_0^2] = \mu_2$ , express  $E[X_t]$  and  $E[X_t^2]$  in terms of  $N, t, a, b, c, d, \mu_1$  and  $\mu_2$ .
3. Suppose that  $X_t$  is a birth and death process on  $\{0, 1, 2\}$  with  $\lambda(s) = 4 - s^2$  and  $\delta(s) = s$ . Find the transition matrix  $P_t$  of  $X_t$ .
4. Suppose that  $X_t$  is a continuous time Markov chain on  $\{1, 2, \dots, N\}$  with rate matrix  $\Lambda$ . Suppose that  $\vec{v} \in R^N$  and  $a$  are an eigenvector/eigenvalue pair for  $\Lambda$ , that is,  $\Lambda\vec{v} = a\vec{v}$ . Let  $v(s)$  be the  $s^{\text{th}}$  coordinate of  $\vec{v}$  and define  $V_t = v(X_t)$ . Find  $E[V_{t+h}|X_t = a]$ .
5. A store owner is trying to decide between two ways of checking out customers. The first possibility is that he can invest in a high-tech cash register that will allow a single cashier to serve 20 customers per hour. The second possibility is that he can hire two cashiers to serve customers, each cashier working at the rate of 10 customers per hour. If customers appear to be checked out at the rate of 15 per hour, compare the two methods of serving the customers by computing the mean and variance of the number of customers waiting to be checked out.