Homework 8: Birth and Death Models

One point problems

1. Suppose that the state space, S, for a birth and death process X_t is $\{0, 1, 2, 3, 4\}$, and that $\lambda(s) = 4 - s$ while $\delta(s) = s$. Find explicitly the limiting distribution and compute its mean and variance.

2. (continuation) Suppose that the initial distribution of this birth and death process is the uniform distribution on S. Find a differential equation for $\mathrm{E}[X_t]$ and solve it. Verify that the limiting value of $\mathrm{E}[X_t]$ is the mean of the limiting distribution.

3. (continuation) Find a differential equation for $\mathrm{E}[X_t^2]$ and solve it. Use this to give the variance for X_t, and confirm that its limit is the variance of the limiting distribution.

4. Suppose that the state space, S, for a birth and death process X_t is $\{0, 1, 2, \ldots, 20\}$, and that $\lambda(s) = 3(20 - s) + 2(20 - s)^2$ while $\delta(s) = 3s + 2s^2$. Suppose that $\Pr(X_0 = 10) = 1$. Find a differential equation for $\mathrm{E}[X_t]$ and solve it.

5. (continuation) Find a differential equation for $\mathrm{E}[X_t^2]$ and solve it. Use this to give the variance for X_t.

6. (continuation) What are the limiting mean and variance for X_t?

7. Let X_t be a birth and death process whose state space is the non-negative integers. Suppose that $\delta(s) = 2s$ and $\lambda(s) = s + 1$. Give an explicit formula for stationary distribution of this birth and death process, and compute the limiting mean and variance.

8. Let X_t be a birth and death process whose state space is the non-negative integers. Suppose that $\delta(s) = s \cdot \lambda(s - 1)/\delta$, that is $\delta(1) = \lambda(0)/\delta$, $\delta(2) = 2\lambda(1)/\delta$, and so on. Find the limiting distribution for this birth and death process.

9. A queue with two servers. Suppose that one line of customers is served by two cashiers. The customers arrive at rate λ no matter how many people are in the system. There are two cashiers who each serve customers at the rate μ. If there are customers waiting and a customer finishes service a new customer takes his place. Let (x, y) be the state of the system when x customers are in line and y customers are in service. Create a state diagram showing the rates at which the system changes from one state to another.

10. (continuation) Find the limiting distribution for this model assuming that $\lambda < 2\mu$. Explain why it should be obvious that this condition should be needed.

Two point problems

11. A birth and death process with state space $\{0, 1\}$ is called a **Telegraph process**. Let $\lambda(0) = \lambda$ and $\delta(1) = \delta$. Give an explicit formula for the transition matrix P_t.

12. Consider the pure death process X_t on $\{1, \ldots, N\}$ with $\delta(s) = s(s - 1)/2$ and $\Pr(X_0 = N) = 1$. Derive and solve the differential equations for $\mathrm{E}[X_t]$ and $\mathrm{E}[X_t^2]$, and give a formula for $\mathrm{Var}[X_t]$. In this model the rate of death is proportional to the number of possible pairs of objects present in the system. Imagine that objects collide at random, and when they do, the coalesce into a single object.

13. Constructing a birth and death process with arbitrary limiting distribution. Suppose that X_t is a birth and death process on the non-negative integers, and

$$\sum_{s=0}^{\infty} p(s) = 1$$

where $p(s) > 0$ for all $s \geq 0$ and $p(-1) = 0$. Find $\lambda(s)$ and $\delta(s)$ so that the limiting distribution of X_t is $p(\cdot)$, that is, for each s, $\lim_{t\to\infty} \Pr(X_t = s) = p(s)$.

This shows that we can have a birth and death process with $\Pr(X_t = s) = p(s)$ for all $t \geq 0$ and any probability distribution we want.

14. If X_t is a birth and death process on S and $c : S \to [0, \infty)$. Interpret $c(X_t)$ is a cost we incur at time t, and the total discounted cost we incur is

$$C := \int_0^\infty c(X_t) e^{-rt} dt.$$

If $S = \{0, 1, 2, 3, 4\}$, $\delta(s) = s(5 - s)$ and $\lambda(s) = (s + 1)(4 - s)$, and $c(u) = u(4 - u)$ compute $E[C]$. You may assume that

$$\mathrm{E}[C] = \int_0^\infty \mathrm{E}[c(X_t)] e^{-rt} dt.$$

15. Suppose that Λ is the rate matrix for a finite state continuous time Markov chain. We know that the transition matrix P_t is given by $P_t = \exp(t\Lambda)$. Prove that $\pi P_t = \pi$ if and only if $\pi\Lambda = \vec{0}$.