Homework 6: Exam Revies

There are exactly 16 points of problems.

One point problems

1. Solve for f_n if $f_{n+2} = 4f_{n+1} - 3f_n + n^2$ if $n \in \{0, 1, \ldots, 15\}$, $f_0 = 1$ and $f_{17} = 4$.

2. Suppose the offspring distribution of a branching process is $p_k = (1/4)(3/4)^k$, for $k = 0, 1, \ldots$. Find the probability that this branching process becomes extinct.

3. A quality control inspector classifies each item she inspects as acceptable (A) or unacceptable (U). She notices that on average if the current item she is inspecting is unacceptable half the time the next item is also unacceptable, while if the current item is acceptable, 9 times out to 10 the next item is acceptable as well. Model this as a Markov chain and calculate the long term percentage of unacceptable items.

4. Suppose in a gambler's ruin problem the probability of winning is 1/3 and the probability of losing is 2/3. Assume the total capital in the game is \$10.00 and \$1.00 is gained or lost on each play. Set up the appropriate difference equation for the probability of ruin if the player has capital x, f_x, and solve for f_x. What choice of x makes f_x closest to 1/2?

5. Consider the Markov chain X_n with state space $\{0, 1, 2\}$. Suppose that

$$\begin{aligned} \Pr(X_0 = 0) &= 1/3 \\ \Pr(X_0 = 1) &= 1/6 \\ \Pr(X_0 = 2) &= 1/2 \end{aligned}$$

   and that the transition matrix is

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/6 & 1/6 & 2/3 \end{bmatrix}$$

   Compute $E[X_3]$. and find $\lim_{n\to\infty} \mathrm{E}[X_n]$.

6. Suppose that we have a transition matrix P given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/2 \\ 1/3 & 0 & 1/3 & 1/3 \end{bmatrix}.$$

   Compute $\lim_{n\to\infty} P^n$ exactly and interpret all the entries in the resulting matrix.

7. For the transition matrix in the preceding exercise, compute the expected time in the transient states given that the initial state is state 4.

8. Define **communicating classes**. What is meant by a **class property**? Give some examples of class properties and explain why it is useful to know that a property is a class property.

Two point problems

1. Suppose that X has the standard normal distribution. Use the fact that that for any positive integer n,

$$\mathrm{E}[X^{2n}] = \frac{(2n)!}{2^n n!}$$

and the power series expansion of $\cos(u)$ to show that

$$\mathrm{E}[\cos(tX)] = \exp(-t^2/2).$$

Use this to conclude that for any real number t, $\mathrm{E}[\exp(itX)] = \exp(-t^2/2)$.

2. Suppose that X is a random variable taking values in the non-negative integers and let $P(t)$ denote its probability generating function. Suppose that $\Pr(X = 0) > 0$ and $\Pr(X \geq 2) > 0$. Prove that the equation $P(t) = t$ has no more than two solutions on $[0, 1]$. Hint: Suppose there are three solutions, call them u, v and, of course, 1, where $u < v < 1$. Let $Q(t) = P(t) - t$. Apply the mean value theorem to Q on the interval $[u, v]$ and on the interval $[v, 1]$. Use this to create a new interval $[a, b]$ satisfying $u < a < v < b < 1$ for which $Q'(a) = Q'(b) = 0$. Apply the mean value theorem again, this time to Q', to get a contradiction.

3. Suppose that X_k are iid random variables taking the values $d < 0 < u$ and $0 \leq r \leq u$. Let $Y_0 = 0$ and $Y_n = X_1 + X_2 + \cdots + X_n$. Put $S_n = \exp(Y_n - rn)$. Find an expression for $p = \Pr(X_k = u)$ in terms of d, r and u so that for each positive integer n,

$$\mathrm{E}[S_{n+1} | Y_1, \ldots, Y_n] = S_n.$$

Hint: If X and Y are independent then $\mathrm{E}[X|Y] = \mathrm{E}[X]$ while if X is a function of Y then $\mathrm{E}[X|Y] = X$.

4. Suppose that we consider the gambler's ruin problem with capital N and the gambler wins 1 unit with probability p, loses one unit with probability q and neither wins nor loses with probability r, where $p + q + r = 1$. Let M_x denote the expected duration of the game if he starts with x units. Explain why $M_x = 1 + pM_{x+1} + rM_x + qM_{x-1}$ and find a formula for M_x and verify that it is correct. Assume p and q not 0. Beware of the case $p = q$.