Mean and Variance Calculation for Finite State Birth and Death Processes

We give two related approaches to finding the mean and variance of a continuous time birth and death process with finite state space. The basic set up is that the state space S is $S = \{0, 1, 2, \ldots, N\}$ for some $N \geq 1$, that

$$\begin{aligned}
\Pr(X_{t+h} = s+1 | X_t = s) &= \lambda(s)h + o(h) \\
\Pr(X_{t+h} = s-1 | X_t = s) &= \delta(s)h + o(h) \\
\Pr(X_{t+h} = s | X_t = s) &= 1 - (\lambda(s) + \delta(s))h + o(h)
\end{aligned}$$

We assume that $\delta(s) > 0$ if $s \neq 0$ and $\delta(0) = 0$, while $\lambda(s) > 0$ if $s \neq N$ and $\lambda(N) = 0$. We are particularly intereested in the case where $\lambda(\cdot)$ and $\delta(\cdot)$ are polynomials of degree at most two, for then we can outline a direct method for computing $\mathrm{E}[X_t]$ and $\mathrm{E}[X_t^2]$ without computing $\Pr(X_t = s)$.

Remember that

$$\begin{aligned}
\mathrm{E}[X_t] &= \sum_{s \in S} s \Pr(X_t = s) \\
\mathrm{E}[X_t^2] &= \sum_{s \in S} s^2 \Pr(X_t = s)
\end{aligned}$$

Since $P_t = \exp(t\Lambda)$ we know that both $\mathrm{E}[X_t]$ and $\mathrm{E}[X_t^2]$ are differentiable functions of t.

1 Differential Equation Method

We will construct differential equations for the mean and second moment of X_t.

1.1 First Moment

Suppose that $\lambda(s) - \delta(s) = a - bs$. Since $\lambda(N) = \delta(0) = 0$ we have

$$\begin{aligned}
\lambda(0) &= a \\
-\delta(N) &= a - bN
\end{aligned}$$

so

$$\begin{aligned}
a &= \lambda(0) \\
b &= \frac{\lambda(0) + \delta(N)}{N}.
\end{aligned}$$

In particular, both a and b are positive.

Since there are only a finite number of states, all the $o(h)$ terms will always recombine to a single $o(h)$ term.

Therefore,

$$\begin{aligned}
\mathrm{E}[X_{t+h} | X_t = s] &= \sum_{k=-1}^{1} (s+k) \Pr(X_{t+h} = s+k | X_t = s) + o(h) \\
&= s - \Pr(X_{t+h} = s-1 | S_t = s) + \Pr(X_{t+h} = s+1 | X_t = s) + o(h) \\
&= s + (\lambda(s) - \delta(s))h + o(h) \\
&= s + (a - bs)h + o(h) \qquad (1)
\end{aligned}$$

Put $M_1(t) = \mathrm{E}[X_t]$. It follows from (1) that

$$\begin{aligned} M_1(t+h) &= \sum_{s\in S} \mathrm{E}[X_{t+h}|X_t = s]\,\Pr(X_t = s) \\ &= \sum_{s\in S}(s + (a-bs)h)\,\Pr(X_t = s) + o(h) \\ &= \sum_{s\in S} s\,\Pr(X_t = s) + h\sum_{s\in S}(a-bs)\,\Pr(X_t = s) + o(h) \\ &= M_1(t) + ah - bhM_1(t) + o(h) \qquad (2) \end{aligned}$$

Let $h \to 0^+$ and (since M_1 is differentiable) we get

$$M_1'(t) + bM_1(t) = a$$

which has as its solution

$$M_1(t) = \left(M_1(0) - \frac{a}{b}\right)\exp(-bt) + \frac{a}{b} \qquad (3)$$

In particular, as $t \to \infty$ we have $M_1(t)$ converging to a/b, and

$$\frac{a}{b} = \frac{N\lambda(0)}{\lambda(0) + \delta(N)}.$$

1.2 Second Moment

Suppose then that we want to extend this method to second moments. This will require that $\lambda(s) - \delta(s)$ is a polynomial of degree 1 (or less) in s while $\lambda(s) + \delta(s)$ is a polynomial of degree 2 (or less) in s. This forces both $\lambda(s)$ and $\delta(s)$ to be polynomials of degree 2 (or less) in s, and, if both are quadratic, they have to have the same leading coefficient. Hence we must have

$$\begin{aligned} \delta(s) &= As + Cs^2 \\ \lambda(s) &= B(N-s) + C(N-s)^2 \end{aligned}$$

This means that there are constants a, b, c, d expressible in terms of A, B, C, N so that

$$\begin{aligned} \lambda(s) - \delta(s) &= a - bs \\ \lambda(s) + \delta(s) &= a - ds + cs^2 \end{aligned}$$

We now calculate in a manner parallel to what we did for $M_1(t)$.

$$\begin{aligned} \mathrm{E}[X_{t+h}^2|X_t = s] &= \sum_{k=-1}^{1}(s+k)^2\,\Pr(X_{t+h} = s+k|X_t = s) + o(h) \\ &= \sum_{k=-1}^{1}(s^2 + 2ks + k^2)^2\,\Pr(X_{t+h} = s+k|X_t = s) + o(h) \\ &= s^2 - 2s(\lambda(s) - \delta(s))h + (\lambda(s) + \delta(s))h + o(h) \\ &= s^2 - 2s(a - bs)h + (a - ds + cs^2)h + o(h) \\ &= s^2 + (2b + c)s^2h - (d + 2a)sh + ah + o(h) \end{aligned}$$

Therefore

$$\begin{aligned}
M_2(t+h) &= \sum_{s\in S} \mathrm{E}[X_{t+h}^2|X_t = s]\Pr(X_t = s) \\
&= \sum_{s\in S}(s^2 + (2b+c)s^2h - (2a+d)sh + ah + o(h))\Pr(X_t = s) \\
&= M_2(t) + (2b+c)M_2(t)h - (2a+d)M_1(t)h + ah + o(h) \qquad (4)
\end{aligned}$$

Therefore
$$\frac{M_2(t+h) - M_2(t)}{h} = (2b+c)M_2(t) - (2a+d)M_1(t) + a + \frac{o(h)}{h}.$$

Letting $h \to 0^+$ we see that

$$M_2'(t) = (2b+c)M_2(t) - (2a+d)M_1(t) + a. \qquad (5)$$

Since $M_1(t)$ is known we may solve this differential equation as well by integrating factors.

1.3 Parameter restrictions

We know from the discussion in the previous section that we must have

$$\begin{aligned}
2\lambda(s) &= 2a + (b+d)s + cs^2 \\
2\delta(s) &= (d-b)s + cs^2.
\end{aligned}$$

We also must have

$$\begin{aligned}
2\lambda(s) &> 0 \text{ if } s < N \\
2\lambda(N) &= 0 \\
2\delta(s) &> 0 \text{ if } s > 0
\end{aligned}$$

so it is convenient to write

$$\begin{aligned}
\lambda(s) &= A(N-s) + C(N-s)^2 \\
\delta(s) &= Bs + Cs^2.
\end{aligned}$$

We can, by comparing coefficients, translate from a, b, c, d to A, B, C. First, we will explore the restrictions on A, B, C.