Continuous Time Markov Chain Basics

1 The Transition Matrix

As earlier, we presume a set of states S that is either finite or countably infinite. We have a set of random variables $\{X_t, t \geq 0\}$ satisfying the Markov Property, namely for every non-negative integer k, $0 \leq t_0 < t_1 < \cdots < t_k < t$ and $h > 0$ we have

$$\Pr(X_{t+h} = y | X_t = x, X_{t_k} = x_k, \ldots, X_{t_0} = x_0) = \Pr(X_{t+h} = y | X_t = x) =: P_t(x, y).$$

Just as in the discrete time case, we have the Chapman-Kolmogorov equations, which in matrix form say

$$P_{t+s} = P_t P_s.$$

We assume that the time elapsed between transitions to new states are independent exponential random variables, each with a rate depending on the state currently occupied. We then have

$$\Pr(X_{t+h} = x | X_t = x) = 1 - \lambda_x h + o_{x,x}(h)$$

while for $y \neq x$ we have

$$\Pr(X_{t+h} = y | X_t = x) = p_{x,y} \lambda_x h + o_{x,y}(h)$$

In matrix form we have

$$P_h = I + h\Lambda + A_n$$

with

$$\Lambda(x, y) = \begin{cases} -\lambda_x & \text{if} \quad y = x \\ p_{x,y}\lambda_x & \text{if} \quad y \neq x \end{cases}$$

and

$$A_h(x, y) = o_{x,y}(h).$$

If M is a matrix we define

$$\|M\| = \max\{\sum_{y \in S} |M(x, y)|, x \in S\}$$

and we assume that $\|A_h\|$ is $o(h)$ and $\|\Lambda\| < \infty$. (This is not an additional assumption if S is finite.)

If this is the case then it can be shown that since $P_t = (P_{t/n})^n$ for every n that

$$\begin{aligned} P_t &= \lim_{n\to\infty} (P_{t/n})^n \\ &= \lim_{n\to\infty} (I + (t/n)\Lambda + A_{t/n})^n \\ &= \lim_{n\to\infty} (I + (t/n)\Lambda)^n \\ &= \exp(t\Lambda). \end{aligned}$$

The matrix Λ is called the rate matrix for the Markov Chain.

2 Stationary Probabilities

We should like to solve $\vec{\pi} = \vec{\pi} P_t$ independent of t. However, since

$$\exp(t\Lambda) = I + \sum_{k=1}^{\infty} \frac{t^k}{k!} \Lambda^k$$

$\vec{\pi} = \vec{\pi} P_t$ is the same as

$$\vec{0} = \sum_{k=1}^{\infty} \frac{t^k}{k!} \pi \Lambda^k$$

If we divide both sides of this equation by t and let $t \to 0^+$ we see that we must have $\vec{0} = \vec{\pi}\Lambda$, while if this is the case then we will have $\vec{\pi} = \vec{\pi} P_t$ for any $t \geq 0$.

The equation $\vec{\pi}\Lambda = \vec{0}$ is called the rate balancing equation.

3 Birth and Death Models

Suppose now that $S = \{0, 1, \ldots\}$. If $p_{x,y} = 0$ for $|x - y| \neq 1$ the Markov Chain is called a Birth and Death Model. In such models it is usual to redefine $p_{x,x+1}\lambda_x$ to be λ_x, $p_{x,x-1}\lambda_x$ to be δ_x and λ_x is replaced by $\lambda_x + \delta_x$. $\delta_0 = 0$. This means that Λ takes the form

$$\begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & & & & \\ \delta_1 & -\delta_1 - \lambda_1 & \lambda_1 & 0 & 0 & \cdots & & \\ 0 & \delta_2 & -\delta_2 - \lambda_2 & \lambda_2 & 0 & 0 & \cdots & \\ 0 & 0 & \delta_3 & -\delta_3 - \lambda_3 & \lambda_3 & 0 & 0 & \cdots \\ \vdots & \vdots & \cdots & & & & & \end{bmatrix}$$

If we write out the rate balancing equations we see that

$$\begin{aligned} -\lambda_0\pi_0 + \delta_1\pi_1 &= 0 \\ \lambda_0\pi_0 - (\lambda_1 + \delta_1)\pi_1 + \delta_2\pi_2 &= 0 \\ \lambda_1\pi_1 - (\lambda_2 + \delta_2)\pi_2 + \delta_3\pi_3 &= 0 \\ \lambda_{x-1}\pi_{x-1} - (\lambda_x + \delta_x)\pi_x + \delta_{x+1}\pi_{x+1} &= 0 \end{aligned}$$

By sequentially adding each equation to the following equation we come to

$$\begin{aligned} \delta_1\pi_1 - \lambda_0\pi_0 &= 0 \\ \delta_2\pi_2 - \lambda_1\pi_1 &= 0 \\ \delta_3\pi_3 - \lambda_2\pi_2 &= 0 \\ \delta_{x+1}\pi_{x+1} - \lambda_x\pi_x &= 0 \end{aligned}$$

which means that if $\delta_x > 0$ for $x \geq 1$ that

$$\pi_{x+1} = \frac{\lambda_x}{\delta_{x+1}}\pi_x = \prod_{k=0}^{x} \frac{\lambda_k}{\delta_{k+1}}\pi_0$$

3.1 The M|M|1 Queuing System

Here is the simplest example: $\delta_x = \delta$, $\lambda_x = \lambda$. Then

$$\pi_x = \left(\frac{\lambda}{\delta}\right)^x \pi_0.$$

If $\delta > \lambda$ we can sum both sides from 0 to ∞ to get

$$1 = \frac{\delta}{\delta - \lambda}\pi_0$$

so

$$\pi_x = \left(1 - \frac{\lambda}{\delta}\right)\left(\frac{\lambda}{\delta}\right)^x,$$

a geometric distribution.