A stock price model

1 Geometric Brownian Motion

Suppose now that B_t is standard Brownian motion. Assume that for the time period in question it is possible to get a guarenteed interest rate of $100\alpha\%$ per year. This means that if S_t is the price of a share of stock at time t, then the current value of a share of stock is $\exp(-\alpha t)S_t$. We propose to model the share price S_t by $S_0 \exp(\sigma B_t + \mu t)$ for some $\sigma > 0$ and $\mu \geq 0$. The process S_t is called **geometric Brownian motion**.

We need to know under what conditions is the discounted stock price, $\exp(-\alpha t)S_t$ a fair price process, that is, when is it the case that

$$\mathrm{E}[\exp(-\alpha(t+s))S_{t+s}|B_t = b] = \exp(-\alpha t)S_0 \exp(\sigma b + \mu t)?$$

So, we calculate:

$$\begin{aligned} & \mathrm{E}[\exp(-\alpha(t+s))S_{t+s}|B_t = b] \\ = \; & \exp(-\alpha t - rs)\mathrm{E}[S_0 \exp(\sigma(B_{t+s} - B_t) + \mu s + \sigma B_t + \mu t)|B_t = b] \\ = \; & S_0 \exp(-\alpha t + \mu t + \sigma b)E[\exp((\mu - \alpha)s + \sigma(B_{t+s} - B_t))|B_t = b] \\ = \; & S_0 \exp(-\alpha t + \mu t + \sigma b)E[\exp((\mu - \alpha)s + \sigma(B_{t+s} - B_t))] \end{aligned}$$

so we would like to have

$$E[\exp((\mu - \alpha)s + \sigma(B_{t+s} - B_t))] = 1.$$

The random variable $B_{t+s} - B_t$ is normally distributed with mean 0 and variance s, so it has the same distribution as $\sqrt{s}N$ where N has the standard normal distribution. This means

$$\begin{aligned} E[\exp((\mu - \alpha)s + \sigma(B_{t+s} - B_t))] &= \exp((\mu - \alpha)s)E[\exp(\sigma\sqrt{s}N)] \\ &= \exp((\mu - \alpha)s)\exp(s\sigma^2/2) \\ &= \exp((\mu + \frac{\sigma^2}{2} - \alpha)s) \end{aligned}$$

so we shall need

$$\mu + \frac{\sigma^2}{2} = \alpha$$

to have the value of future shares equal to the present value on average.

1.1 Option value

Suppose then that $K > 0$ is given and represents the strike price of a European option on our stock. That is, we have an agreement that you will sell me a share of this stock at a time T in the future, where T is measured in units of years. Typically, $T = 1/4$, or about 90 days. My profit on the stock is 0 if $S_T \leq K$ and is $S_T - K$ if $S_T \geq K$. In other words, my profit is $\max(0, S_T - K)$, which we will write as $(S_T - K)^+$. The current value of this profit is

$\exp(-\alpha T)(S_T - K)^+$. Our task is to calculate $\exp(-\alpha T)E[(S_T - K)^+]$. To do this, let N be a standard normal random variable.

$$
\begin{aligned}
\mathrm{E}[(S_T - K)^+] &= \mathrm{E}[(S_0 \exp(\sigma B_T + \mu T) - K)^+] \\
&= \mathrm{E}[(S_0 \exp(\sigma \sqrt{T} N + \mu T) - K)^+] \\
&= \int_L^\infty S_0 \exp(\sigma \sqrt{T} z + \mu T) - K \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz \\
&= \int_L^\infty S_0 \exp(\sigma \sqrt{T} z + \mu T) \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz - K \int_L^\infty \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz
\end{aligned}
$$

where

$$
L = \frac{\log(K) - \log(S_0) - \mu T}{\sigma \sqrt{T}}.
$$

Let $\Psi(s)$ be the survival function for the standard normal distribution:

$$
\Psi(s) = \int_s^\infty \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz.
$$

and let $\Phi(t) = 1 - \Psi(t)$ be the corresponding distribution function. Remember that by symmetry $\Psi(t) = \Phi(-t)$.

We have

$$
\mathrm{E}[(S_T - K)^+] = \int_L^\infty S_0 \exp(\sigma \sqrt{T} z + \mu T) \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz - K\Psi(L)
$$

The remaining integral can also be expressed in terms of Ψ:

$$
\begin{aligned}
&\int_L^\infty S_0 \exp(\sigma \sqrt{T} z + \mu T) \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz \\
&= S_0 \exp(\mu T) \frac{1}{\sqrt{2\pi}} \int_L^\infty \exp\left(\sigma \sqrt{T} z - \frac{z^2}{2}\right) dz \\
&= S_0 \exp(\mu T) \exp(\sigma^2 T/2) \frac{1}{\sqrt{2\pi}} \int_L^\infty \exp\left(-\frac{(z - \sigma\sqrt{T})^2}{2}\right) dz \\
&= S_0 \exp(\mu T) \exp(\sigma^2 T/2) \frac{1}{\sqrt{2\pi}} \int_M^\infty \exp(-u^2/2) du \\
&= S_0 \exp(\mu T) \exp(\sigma^2 T/2) \Psi(M)
\end{aligned}
$$

where $M = L - \sigma\sqrt{T}$.

Recalling that $\mu + (\sigma^2/2) - \alpha = 0$ we have

$$
\begin{aligned}
\exp(-\alpha T)\mathrm{E}[(S_T - K)^+] &= S_0 \Psi(M) - \exp(-\alpha T) K \Psi(L) \\
&= S_0 \Phi(-M) - \exp(-\alpha T) K \Phi(-L) \\
L &= \frac{\log(K) - \log(S_0) - \mu T}{\sigma\sqrt{T}} \\
&= \frac{\log(K/S_0) - \alpha T + (\sigma^2/2) T}{\sigma\sqrt{T}} \\
M &= L - \sigma\sqrt{T}
\end{aligned}
$$

These are called the Black-Scholes option pricing formula for European call options.