An introduction to Brownian Motion

Brownian Motion is an attempt to model the accumulation of a large number of small changes over time. We begin our study of Brownian motion by looking at this issue from several perspectives.

1 Brownian Motion as a Limit of Random Walks

Suppose that Y_k are iid random variables with $\Pr(Y_1 = 1) = \Pr(Y_1 = -1) = 1/2$. We would like to rescale time and space in such a way as to model the accumulations of a large number of small changes in a fixed amount of time.

To this end, let Δx and Δt be fixed positive numbers, where Δt represents the length of time between changes and Δx represents the magnitude of each change. Define

$$X(t; \Delta t, \Delta x) = \sum_{k=1}^{[t/\Delta t]} Y_k \Delta x$$

for each $t \geq 0$. As a function of t, for each ω, $X(t; \Delta t, \Delta x)$ is a step function, where steps occur every Δt units of time and steps are of magnitude Δx.

For example, if $\Delta t = 1/100$, $\Delta X = 1/10$ and $t = 5$ then

$$X(5; 1/100, 1/10) = \sum_{k=1}^{500} \frac{Y_k}{10}.$$

Observe that $\mathrm{E}[X(5; 1/100, 1/10)] = 0$ and $\mathrm{Var}[X(5; 1/10, 1/100)] = 5$, and by the Central Limit Theorem the distribution of $X(5, 1/100, 1/10)$ is approximately normal with mean 0 and variance 5.

Note that as a function of t, X is an independent increments process. Increments are not stationary owing to the fact that $[t+s]$ may not equal $[t]+[s]$. To get an idea of the distribution of an increment we will compute its characteristic function. Since any increment is the sum of iid random variables $Y_k \Delta x$ we first compute the characteristic function of these summands:

$$\begin{aligned} \mathrm{E}[\exp(i\theta Y_k \Delta x)] &= \frac{1}{2}\exp(i\theta\Delta x) + \frac{1}{2}\exp(-i\theta\Delta x) \\ &= \cos(\theta\Delta x). \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{E}[\exp(i\theta(X(t+s; \Delta t, \Delta x) - X(t; \Delta t, \Delta x)))] &= (\mathrm{E}[\exp(i\theta\Delta x Y_1)])^{[(t+s)/\Delta t]-[t/\Delta t]} \\ &= \cos(\theta\Delta x)^{[(t+s)/\Delta t]-[t/\Delta t]} \end{aligned}$$

We want to see what happens as Δt and Δx converge to zero. Note first that

$$\lim_{\Delta t \to 0^+} \frac{\Delta t}{s}\left(\left[\frac{t+s}{\Delta t}\right] - \left[\frac{t}{\Delta t}\right]\right) = 1$$

so we need to look at

$$(\cos(\theta\Delta x))^{s/\Delta t} = \left(1 - \frac{(\theta\Delta x)^2}{2} + o((\Delta x)^2)\right)^{s/\Delta t}. \tag{1}$$

Recall that

$$\lim_{n\to\infty}\left(1+\frac{u}{n}+o(1/n)\right)^n = \exp(u).$$

In (1) let $s/\Delta t = n$ and $(\Delta x)^2 = c\Delta t$ where $c > 0$. Then

$$\begin{aligned}\left(1-\frac{(\theta\Delta x)^2}{2}+o((\Delta x)^2)\right)^{s/\Delta t}. &= \left(1-\frac{\theta^2 c\Delta t}{2}+o(\Delta t)\right)^n \\ &= \left(1-\frac{\theta^2 cs}{2n}+o(1/n)\right)^n\end{aligned}$$

which converges to $\exp(-\theta^2 cs/2)$ as $n\to\infty$, the characteristic function of a normal distribution with mean 0 and variance cs. Ordinarily we would take $c = 1$, that is $\Delta x = \sqrt{\Delta t}$.

Hence, in the sense of distribution functions,

$$\lim_{\Delta t\to 0^+} X(t;\Delta t,(\Delta t)^2)$$

is a stochastic process $\{B(t), t \ge 0\}$ which has stationary, independent increments, and the distribution of the increment $B(t+s)-B(t)$ is normal with mean zero and variance s.

What is not clear at all from the proceeding is whether for any particular ω, and any particular t,

$$X(t;\Delta t,(\Delta t)^2)$$

converges as Δt approaches zero from above.

2 A Markov chain perspective

To this point we have only considered Markov chains where the state space is countable. Now let us consider the possibility that the state space is the entire real line. In discrete time we have many examples of this. All we need consider is sums of iid random variables where the summands have continuous distributions. Continuous time is another story. However, we can approach this problem from the transition function point of view. In other words, we can try to require that for times $t > s$ and each pair of real numbers x and y,

$$\Pr(X_t \le y | X_s = x) = \int_{-\infty}^{y} f(t,u;s,x)\,du$$

where $f(t,u;s,x)$ is the conditional density of X_t given that $X_s = x$. This conditional density plays the role of the transition function in the Markov chains we have studied to date. One of the simplest examples we can consider is normal distributions. We take

$$f(t,u;s,x) = \frac{1}{\sqrt{2|t-s|\pi}}\exp\left(\frac{(u-x)^2}{2|t-s|}\right).$$

In other words, the distribution of X_t given that $X_s = x$ is a normal distribution with mean x and variance $|t-s|$. It is as if $X_t = X_s + N$ where N and X_s are independent, and N is normal with mean 0 and variance $|t-s|$.

To continue in this vein, we are assuming that we have a stationary transition function so we have to check that the Chapman-Kolmogorov equations hold:

$$\int_{-\infty}^{\infty} f(s+r,u;r,x)f(s+r+t,v;s+r,u)\,du = f(r+s+t,v;r,x)$$

To show this, we work from left to right:

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(s+r,u;r,x)f(s+r+t,v;s+r,u)\,du \\
=\ &\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{\sqrt{st}}\exp\left(-\frac{(u-x)^2}{2s}-\frac{(v-u)^2}{2t}\right)\,du \\
=\ &\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{\sqrt{st}}\exp\left(-\frac{((s+t)u-(xt+vs))^2}{2(s+t)(st)}-\frac{(x-v)^2}{2(s+t)}\right)\,du
\end{aligned}$$

To evaluate this integral, let

$$\begin{aligned}
z &= \frac{(s+t)u-(xt+vs)}{\sqrt{(s+t)st}} \\
dz &= \frac{\sqrt{s+t}}{\sqrt{st}}du \\
\frac{dz}{\sqrt{s+t}} &= \frac{du}{\sqrt{st}}
\end{aligned}$$

which transforms the integral to

$$\begin{aligned}
&\exp\left(-\frac{(x-v)^2}{2(s+t)}\right)\int_{-\infty}^{\infty}\frac{1}{2\pi}\frac{1}{\sqrt{s+t}}\exp\left(-\frac{z^2}{2}\right)\,dz \\
=\ &\frac{1}{\sqrt{2|s+t|}}\exp\left(-\frac{(x-v)^2}{2(s+t)}\right) \\
=\ &f(r+s+t,v;r,x)
\end{aligned}$$

showing that

$$\int_{-\infty}^{\infty} f(s+r,u;r,x)f(s+r+t,v;s+r,u)\,du = f(r+s+t,v;r,x)$$

2.1 Other possibilities

The normal distribution is not the only distribution that can be used for this construction. For example, if we take $s < t$ and $y > x$ we may take

$$f(t,y;s,x) = \frac{1}{\Gamma(t-s)}(y-x)^{t-s-1}\exp(-(y-x))$$

and $f(t,y;s,x) = 0$ if $y \leq x$. Then for $z > x$

$$\int_{x}^{z} f(r+s,y;r,x)f(r+s+t,z;r+s,y)\,dy$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(s)\Gamma(t)} \int_x^z (y-x)^{s-1}(z-y)^{t-1} \exp(-(y-x)-(z-y))\, dy \\
&= \frac{\exp(-(z-x))}{\Gamma(s)\Gamma(t)} \int_x^z (y-x)^{s-1}(z-y)^{t-1}\, dy \\
&= \frac{\exp(-(z-x))}{\Gamma(s)\Gamma(t)} \int_0^{z-x} q^{s-1}(z-x-q)^{t-1}\, dq \\
&= \frac{\exp(-(z-x))}{\Gamma(s)\Gamma(t)} \int_0^1 (v(z-x))^{s-1}((z-x)(1-v))^{t-1}(z-x)\, dv \\
&= \frac{\exp(-(z-x))}{\Gamma(s)\Gamma(t)} (z-x)^{s+t-1} \int_0^1 v^{s-1}(1-v)^{t-1}\, dv \\
&= \frac{\beta(s,t)}{\Gamma(s)\Gamma(t)} (z-x)^{s+t-1} \exp(-(z-x)) \\
&= \frac{1}{\Gamma(s+t)} (z-x)^{s+t-1} \exp(-(z-x)) \\
&= f(r+s+t, z; r, x)
\end{aligned}
$$

There are many other distributions for which this is possible, including the Cauchy distribution.

3 Defining Brownian Motion by Properties

Recall that one way to define the Poisson Process was to say that it was a process with stationary, independent increments and that the increments had Poisson distributions. We were able to prove that there was such a process if there were independent, identically distributed sequences of exponential random variables. The situation for Brownian Motion is more complicated. We can make the following definition.

Definition *A stochastic process* $\{B(t), t \geq 0\}$ *is called standard* **Brownian Motion** *if it has stationary, independent increments and the distribution of the increment* $B(t+s) - B(t)$ *is normal with mean zero and variance* s*.*

The problem with this definition is that no such stochastic process may exist. In fact, one important theorem proves that there is a stochastic process which is Brownian Motion.

To prove this theorem, one constructs a sample space whose elements are functions, a sigma algebra on this space, and then a probability measure, where the probability that a sample point is a continuous function is 1. If ω is a sample point, we put $B_t(\omega) = \omega(t)$, and construct the measure so that B is Brownian Motion.

In any event, two important properties of Brownian motion are

- With probability 1, Brownian Motion is continuous. That is

$$\Pr(\omega : B(t,\omega) \text{ is continuous on } [0,T]) = 1;$$

- With probability 1, Brownian Motion is nowhere differentiable. That is

$$\Pr(\omega : B(t,\omega) \text{ is not differentiable at any } t \in [0,T]) = 1;$$

4 The Order of Magnitude of Brownian Increments

Just how big is $B(t+h) - B(t)$? Given our assertion that we cannot differentiate $B(t)$, this is a natural question. The claim is that

$$\frac{(B(t+h) - B(t))^2}{h} \approx 1.$$

To make this precise, first observe the quantity in question has a chi-square distribution with one degree of freedom, so if we put

$$Y_{t,h} = \frac{(B(t+h) - B(t))^2}{h} - 1 = \frac{(B(t+h) - B(t))^2 - h}{h}$$

we see that

$$\mathrm{E}[Y_{t,h}] = 0$$

and

$$\mathrm{Var}[Y_{t,h}] = 2.$$

The sense in which our assertion is true is in the sense of integration. Specifically,
Lemma: *Suppose that $f : [a,b] \to (-\infty, \infty)$. For each positive integer N let*

$$(x_0^{(N)}, x_1^{(N)}, \ldots, x_N^{(N)})$$

be a partition of $[a,b]$ and let $(s_1^{(N)}, s_2^{(N)}, \cdots, S_N^{(N)})$ satisfy

$$x_{k-1}^{(N)} \leq s_k^{(N)} \leq x_k^{(N)}$$

for $k \in \{1, 2, \ldots, N\}$. Put

$$\begin{aligned} \Delta B_k^{(N)} &= B(x_k^{(N)}) - B(x_{k-1}^{(N)}) \\ \Delta x_k^{(N)} &= x_k^{(N)} - x_{k-1}^{(N)} \\ \mu^{(N)} &= \max_{1 \leq k \leq N} \{x_k^{(N)} - x_{k-1}^{(N)}\}. \end{aligned}$$

Then

$$\mathrm{Var}\left(\sum_{k=1}^{N} f(s_k^{(N)})(\Delta B_k^{(N)})^2 - \Delta x_k^{(N)}) \right) \leq 2\mu^{(N)} \sum_{k=1}^{N} (f(s_k))^2 \Delta x_k^{(N)}.$$

Theorem: *Suppose that $f : [a,b] \to (-\infty, \infty)$ is Riemann integrable. Adopt the same notation as in the preceding Lemma. Then*

$$\sum_{k=1}^{N} f(s_k^{(N)})(\Delta B_k^{(N)})^2$$

converges to

$$\int_a^b f(u)\, du$$

in probability, and if

$$\sum_{N=1}^{\infty} \mu^{(N)} < \infty$$

the convergence is with probability 1.

Thus, in the sense of integration, $(\Delta B_k^{(N)})^2$ serves the same purpose as $\Delta x_k^{(N)}$ does, since

$$\lim_{N\to\infty} \sum_{k=1}^{N} f(s_k^{(N)})\Delta x_k^{(N)} = \int_a^b f(u)\, du$$

if f is Riemann integrable on $[a, b]$.