The No-Arbitrage Condition

Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space, where Ω is the set of outcomes, $\mathcal{F}$ is the set of events, and Pr is the assignment of probabilities. Let $\{R_k, k \in K\}$ be real valued random variable, where R_k represents your return on investment opportunity k. For example, R_1 is your return tomorrow on selling a share of IBM stock that is worth \$100.00 today. If the price tomorrow in \$50.00 per share, then $R_1 = -\$50.00$, and if the price tomorrow is \$200.00 then $R_1 = \$100.00$.

To make things simpler, let us suppose that $K = \{0, 1, \ldots, N\}$, that is, you have N opportunities available to you. Your investment strategy, $\vec{S}$, is a vector $[s_1, s_2, \ldots, s_N]$, $s_k \geq 0$ representing how much of each of the investment opportunities you hold. You might think of them as numbers of shares you might buy or sell. Your wealth, W, is then

$$W := \sum_{k=1}^{N} s_k R_k$$

so W is a new random variable. We are interested in when $W > 0$. Since dividing $\vec{S}$ by any positive constant results in dividing W by that same constant, there is no loss in generality in assuming that $s_1 + \cdots + s_N = 1$.

We say that there is an **arbitrage opportunity** if $\Pr(W \geq 0) = 1$ and $\Pr(W > 0) > 0$. In other words, you cannot lose money with strategy S and you have a positive probability of making money.

Now, suppose that the returns, R_k are known, but the probability assignment, Pr, is not. That is to say, for each $\omega \in \Omega$, we know $R_k(\omega)$. Of course, since things are random, we don't know ω, so we only know the potential returns.

Now, to make things even simpler, suppose that $\Omega := \{1, 2, \ldots, m\}$. Then each R_k is a point $r_k \in R^m$ whose i^{th} coordinate is $R_k(i)$. Likewise, the potential values of your wealth are also a point in R^m, and the set of potential wealths is a convex set in R^m:

$$\mathcal{W} := \{\sum_{k=1}^{N} s_k r_k : s_k \geq 0\}.$$

The other interesting set to us is $R^m_+ := \{x \in R^m : x_k > 0, k = 1, \ldots, m\}$. If $W \in R^m_+$ then one's wealth is positive regardless of the random outcomes. R^m_+ is also convex, and there is a theorem, called the **separating hyperplane theorem**, that says that two convex sets in R^m are disjoint if and only if there is a hyperplane $\vec{p} \cdot \vec{x} = c$ such that one set is contained in $\vec{p} \cdot \vec{x} \leq c$ and the other is contained in $\vec{p} \cdot \vec{x} > c$. For R^m_+ to be contained in $\vec{p} \cdot \vec{x} > c$ $\vec{p}$ cannot have coordinates of different signs, so, without loss of generality we may assume that they are all non-negative, and that they sum to 1. This makes $c \leq 0$ and says that if $\Pr(\{i\}) = \vec{p}_i$ then $\mathrm{E}[R_k] \leq c \leq 0$ for each k. In other words, either there is an arbitrage opportunity, or there is a way to set a probability measure on Ω so that no investment opportunity has positive expected value.

An option is one possible investment opportunity. Suppose that today's price of a share of WEC is \$49.00 and the possible prices tomorrow are \$47.00 and \$50.00. One investment opportunity is to buy a share today and sell it tomorrow. Thus $r_1 = (-2, 1)$. Another opportunity is to do the reverse, sell a share today and buy one tomorrow, which gives $r_2 = (2, -1)$. A third possibility is to buy a call option on a share of IBM to buy a share tomorrow for \$49.50. If you pay c dollars for this option, then $r_3 = (-c, 1/2 - c)$. Suppose there is to be no arbitrage. Then for $s_1 + s_2 + s_3 = 1$ and $s_i \geq 0$ we must have $s_1(-2, 1) + s_2(2, -1) + s_3(-c, 1/2 - c) \leq (0, 0)$.