Vector Spaces: Ordered Bases and Coordinates

All material from Chapter 2 of Linear Algebra by Hoffman and Kunze.

Definition: Let V be a finite dimensional vector space over F. A finite sequence

$$B = (\beta_1, \ldots, \beta_n)$$

elements of V is said to be **an ordered basis for** V if $\{\beta_1, \ldots, \beta_n\}$ is a basis for V.

Definition: If $B = (\beta_1, \ldots, \beta_n)$ is an ordered basis for V and

$$\alpha = \sum_{k=1}^{n} b_k \beta_k$$

then the n-tuple $(b_1, \ldots, b_n)$ is called **coordinates of** α **with respect to** B and b_k is called the k^{th} **coordinate of** α **with respect to** B.

Note that since the β_k are linearly independent that coordinates are uniquely determined by the ordered basis.

Definition: If $B = (\beta_1, \ldots, \beta_n)$ is an ordered basis for V and

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then the element of $F^{n\times 1}$ whose $(k, 1)$ entry is b_k is called the **coordinate matrix of** α **with respect to** B and is denoted by $[\alpha]_B$, or just $[\alpha]$ if there is no ambiguity as to the choice of ordered basis B. Note that coordinate matrices are always column matrices.

Theorem 2.7: Suppose that V is a finite dimensional vector space over F and that B and G are ordered bases for V. Then there is a unique $P \in F^{n\times n}$ so that P is invertible and for all $\alpha \in V$

$$[\alpha]_B = P[\alpha]_G.$$

In fact, if $G = (\gamma_1, \ldots, \gamma_n)$ and $B = (\beta_1, \ldots, \beta_n]$ then the k^{th} column of P is $[\gamma_k]_B$ and the k^{th} column of P^{-1} is $[\beta_k]_G$.

Theorem 2.8 Suppose $P \in F^{n\times n}$ is invertible and V is a finite dimensional vector space over F with ordered basis B. Then there is a unique ordered basis G for V so that for each $\alpha \in V$ we have

$$[\alpha]_B = P[\alpha]_B.$$