Vector Spaces and Inner Product Spaces: Lecture 01

All material from Chapter 2 and 8 of Linear Algebra by Hoffman and Kunze.

Definition: A **vector space** consists of the following:

1. a field F of **scalars**;
2. a set V of objects called **vectors**;
3. An operation, called **vector addition**, which associates to each pair of vectors α and β in V a vector $\alpha + \beta \in V$, called the **sum** of α and β in such a way that
   (a) vector addition is commutative;
   (b) vector addition is associative;
   (c) there is a vector $\vec{0} \in V$, called the **zero vector**, such that $\alpha + \vec{0} = \alpha$ for all $\alpha \in V$.
   (d) for each $\alpha \in V$ there is a vector $-\alpha \in V$ so that $-\alpha + \alpha = \vec{0}$.
4. an operation, called **scalar multiplication**, which associates with each scalar $c \in F$ and each vector $\alpha \in V$ a vector $c\alpha \in V$, called the **product** of c and α, in such a way that
   (a) $1\alpha = \alpha$ for each $\alpha \in V$;
   (b) $(c_1 c_2)\alpha = c_1(c_2\alpha)$;
   (c) $c(\alpha + \beta) = c\alpha + c\beta$
   (d) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$.

Example: If F is a subfield of the field G then G is a vector space over F, but F is not a vector space over G unless $F = G$.

Example: If S is any non-empty set and W is a vector space over F, then the set of all functions from S to W is a vector space over F with the usual definition of sum of functions and multiplication of functions by scalars. This vector space is sometimes denoted by W^S. In particular, F^S is a vector space over F.

Definition: A vector space V over F is called an **inner product space** if

1. F is a subfield of the complex numbers.
2. there is an operation, called the **inner product**, which associates to each pair of vectors α and β a scalar $(\alpha|\beta) \in F$ so that
   (a) $(c\alpha + \beta|\gamma) = c(\alpha|\gamma) + (\beta|\gamma)$;
   (b) $(\beta|\alpha)$ is the complex conjugate of $(\alpha|\beta)$.
   (c) $(\alpha|\alpha) > 0$ if $\alpha \neq \vec{0}$.

Example 1: Suppose that $V = F^{n\times 1}$ where F is a subfield of the complex numbers and $w_k > 0$ for $k = 1, 2, \ldots, n$. Then

$$(\alpha|\beta) = \sum_{k=1}^{n} w_k \alpha_k \overline{\beta_k}$$

defines an inner product on V. The scalars w_k are called **weights**. If the weights are all 1 this is called the **standard inner product**. If F is contained in the real numbers and the weights are all 1 this is called the **dot product**.

Example 2: Suppose that V is the set of continuous functions from $[-1, 1]$ into the complex numbers, F is a subfield of the complex numbers, and $w \in V$ and $w(x) > 0$ for $x \in (-1, 1)$ then

$$(\alpha|\beta) = \int_{-1}^{1} \alpha(x)\overline{\beta(x)}w(x)\, dx$$

defines an inner product on V.

Example 3: Suppose that $T : V \to W$ is a one-to-one linear transformation and W is an inner product space with inner product $(\cdot|\cdot)_W$. Then

$$(\alpha|\beta)_V = (T(\alpha)|T(\beta))_W$$

defines an inner product on V.

Proposition: Suppose that V is a vector space over F.

1. $c\vec{0} = \vec{0}$;
2. $0\alpha = \vec{0}$;
3. $c\alpha = \vec{0}$ implies $c = 0$ or $\alpha = \vec{0}$.
4. $-1\alpha = -\alpha$

If V is an inner product space then

1. $(\alpha|c\beta + \gamma) = \overline{c}(\alpha|\beta) + (\alpha|\gamma)$ where $\overline{c}$ denotes the complex conjugate of c.
2. $(\alpha|\alpha) = 0$ if and only if $\alpha = \vec{0}$.

Definition: If V is an inner product space over F then the **norm** of a vector α, denoted by $\|\alpha\|$ is given by

$$\|\alpha\| = \sqrt{(\alpha|\alpha)}.$$

If $\|\alpha\| = 1$ we say that α is a **unit vector**. If F contains the real numbers and $\alpha \neq \vec{0}$ then the vector $(1/\|\alpha\|)\alpha$ is called the **direction** of α. The direction of α is a unit vector.

Polarization identities: Relations between the norm and the inner product.

1. If F is a subfield of the real numbers then

$$4(\alpha|\beta) = (\alpha + \beta|\alpha + \beta) - (\alpha - \beta|\alpha - \beta)$$

so if F is the real numbers then

$$(\alpha|\beta) = \frac{1}{4}\left(\|\alpha + \beta\|^2 - \|\alpha - \beta\|^2\right).$$

2. If F is the complex numbers, then

$$(\alpha|\beta) = \frac{1}{4}\sum_{k=1}^{4} i^k \|\alpha + i^k\beta\|^2.$$

Definition α and β are said to be **orthogonal** if $(\alpha|\beta) = 0$. A set of vectors is said to be **orthogonal** if any pair of vectors in the set is orthogonal. A set of orthogonal vectors is said to be **orthonormal** if each vector in the set is a unit vector.

Lemma: If $\alpha \neq \vec{0}$, define $\text{proj}_\alpha : V \to V$ by

$$\text{proj}_\alpha(\beta) := \frac{(\beta|\alpha)}{(\alpha|\alpha)}\alpha.$$

$\text{proj}_\alpha(\beta)$ is called the **(orthogonal) projection of** β **onto** α, and $\beta - \text{proj}_\alpha(\beta)$ is orthogonal to α.

Pythagorean Theorem: If α and β are orthogonal then

$$\|\alpha\|^2 + \|\beta\|^2 = \|\alpha - \beta\|^2$$

Law of Cosines: If $\alpha \neq \vec{0}$ and $\beta \neq \vec{0}$ then

$$\|\alpha - \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 - 2\|\alpha\|\|\beta\|\frac{((\alpha|\beta) + (\beta|\alpha))/2}{\|\alpha\|\|\beta\|}$$

Theorem 8.1: If V is an inner product space then for any vectors α and β and any scalar c:

Norm scaling: $\|c\alpha\| = |c| \cdot \|\alpha\|$.

Positivity: If $\alpha \neq \vec{0}$ then $\|\alpha\| > 0$.

Cauchy-Schwarz-Bunyakowski Inequality: $|(\alpha|\beta)| \leq \|\alpha\| \cdot \|\beta\|$.

Triangle inequality: $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.

The CSB Inequality is proven by applying the Pythagorean Theorem to $\text{proj}_\alpha(\beta)$ and $\text{proj}_\alpha(\beta) - \beta$.

Definition: Let V be a vector space over the field F. A **subspace** of V is a subset W of V which is itself a vector space over F with respect to the operations of V.

Theorem 2.1: A non-empty subset W of a vector space V over F is a subspace of V if and only if W is closed under scalar multiplication and vector addition, that is, for every pair of vectors α and β in W and each scalar $c \in F$ we have $c\alpha + \beta \in W$.

Example: Let V be an inner product space, and let $\alpha \in V$ be given. $W = \{\beta \in V : (\beta|\alpha) = 0\}$ is a subspace of V.

Theorem 2.2: The intersection of subspaces of V is a subspace of V.

Example: If W is a subset of an inner product space, the set

$$W^\perp := \{\beta \in V : (\beta|\omega) = 0 \text{ for all } \omega \in W\}$$

is a subspace called the **orthogonal complement of W**.

Definition: Let $S \subset V$. The subspace **spanned by** S is the intersection of all subspaces of V that contain S. If S is empty or if $S = \{\vec{0}\}$ then the span of S is $\{\vec{0}\}$.

Definition: If $S \subset V$, a vector β is said to be a **linear combination** of the elements of S if there are scalars $c_1, \ldots, c_n$ and vectors $\sigma_1, \ldots, \sigma_n$ in S so that

$$\beta = \sum_{k=1}^{n} c_k \sigma_k$$

If $\emptyset \neq S \subset V$ we say S is **linearly independent** if the only linear combination of distinct elements of S that equals $\vec{0}$ is the one with all the scalars equal to 0.

Observation: If S is an orthonormal set in V and β is a a linear combination of elements of S then

$$\beta = \sum_{\sigma \in S} (\beta|\sigma)\sigma.$$

Note: Under the hypotheses, $(\beta|\sigma) \neq 0$ for only a finite number of elements of S.

Theorem 8.2: An orthonormal set is a linearly independent set.

Theorem 2.3: The subspace spanned by a non-empty set S of a vector space V is the set of all linear combinations of elements of S.

Definition: If S_1, S_2, …, S_k are subsets of a vector space V, the set of all sums

$$\sigma_1 + \cdots + \sigma_k$$

where $\sigma_j \in S_j$ is called the **sum** of the subsets S_1, S_2,…, S_k and is denoted by

$$S_1 + S_2 + \cdots + S_k$$

or

$$\sum_{j=1}^{k} S_j.$$

Proposition: If W_1, W_2, $\ldots$, W_n are subspaces of V then $W_1 + W_2 + \cdots + W_n$ is the subspace spanned by $W_1 \cup W_2 \cdots \cup W_n$.

Definitions: If $A \in F^{m \times n}$ then the **row vectors** of A are the elements of F^m given by

$$\rho_i = (A_{i,1}, \ldots, A_{i,n})$$

while the **column vectors** of A are the elements of F^n given by

$$\kappa_j = (A_{1,j}, \ldots, A_{m,j})$$

The **row space** of A is the subspace spanned by the row vectors of A, and the **column space** of A is the subspace spanned by the column vectors of A.