Transposes and Adjoints of Linear Transformations

All material from Chapter 3 and 8 of Linear Algebra by Hoffman and Kunze.

Definition: Let V and W be a vector spaces over a field F. Let $T \in L(V, W)$. Then $T^t : W^* \to V^*$ given by $T^t(f) = f \circ T$ is a function called the **transpose of** T.

Theorem 3.21: $T^t \in L(W^*, V^*)$. Furthermore, there is precisely one $S \in L(W^*, V^*)$ so that $S(f) = T^t(f)$ for all $f \in W^*$.

Theorem 3.22: Let V and W be vector spaces over the field F, and let $T \in L(V, W)$. Then $N(T^t) = R(T)^0$ and $R(T^t) \subset N(T)^0$. If V and W are finite dimensional then we also have

1. $\dim(R(T)) = \dim(R(T^t))$
2. $R(T^t) = N(T)^0$.

Demonstration:

$$\begin{aligned}
\dim(R(T)) + \dim(R(T)^0) &= \dim(W) \\
\dim(N(T^t)) + \dim(R(T^t)) &= \dim(W) \\
\dim(R(T)) + \dim(R(T)^0) &= \dim(N(T^t)) + \dim(R(T^t)) \\
\dim(R(T)^0) &= \dim(N(T^t)) \\
\dim(R(T)) &= \dim(R(T^t))
\end{aligned}$$

and

$$\begin{aligned}
\dim(N(T)) + \dim(N(T)^0) &= \dim(V) \\
\dim(N(T)) + \dim(R(T)) &= \dim(V) \\
\dim(N(T))^0 - \dim(R(T)) &= 0 \\
\dim(R(T)) &= \dim(R(T^t)) \\
\dim(N(T)^0) &= \dim(R(T^t))
\end{aligned}$$

Definition: If $A \in F^{m\times n}$ then $A^t \in F^{n\times m}$ is given by $A^t(j,k) = A(k,j)$. A^t is called the **transpose of** A.

Theorem 3.23: Let V and W be finite dimensional vector spaces with ordered bases B and G respectively, and let B^* and G^* be the respecitive dual bases for V^* and W^*. If $[T]$ is the matrix of T with respect to B and G and $[T^t]$ is the matrix of T^t with respect to G^* and B^* then $[T]^t = [T^t]$.

Theorem 3.24: If $A \in F^{m\times n}$ then the row rank of A equals the column rank of A.

Definition: If V is a inner product space and T is a linear operator on V we say that T is a **bounded operator** if there is a constant M_T so that $\|T(\alpha)\| \le M_T\|\alpha\|$ for every $\alpha \in V$. Since T is linear and $\|c\alpha\| = |c|\alpha$, T is a a bounded linear operator if and only if $\|T(\nu)\| \le M_T$ for all unit vectors ν.

Proposition: Every linear operator on a finite dimensional inner product space is bounded.

To see why, let $(\nu_1, \dots, \nu_N)$ be an ordered orthonormal basis for V, and suppose that ν is a unit vector.

$$1 = \|\nu\|^2 = \sum_{k=1}^{N} |(\nu|\nu_k)|^2$$

so $|(\nu|\nu_k)| \le 1$ for all k. Let $M = \max\{\|T(\nu_1)\|, \dots, \|T(\nu_N)\|\}$. Then

$$\|T(\alpha)\| =\le \sum \|(\alpha|\nu_k)|\|T(\nu_k)\| \le M \sum_{k=1}^{N} |(\alpha|\nu_k)| = NM$$

so we can take $M_T = NM$.

Definition: Suppose that V is a inner product space and T is a linear operator on V. If there is a linear operator S on V such that $(T(\alpha)|\beta) = (\alpha|S(\beta))$ for all α and β in V then S is called the **adjoint** of T. (T can have a most one adjoint since if

$$(\alpha|S(\beta)) = (T(\alpha)|\beta) = (\alpha|S'(\beta))$$

for all α then $S(\beta) = S'(\beta)$.) The adjoint of T (if it exists) is denoted by T^*.

Proposition: If T has an adjoint T^* then T is the adjoint of T^*.

This is easy:

$$(T^*(\alpha)|\beta) = \overline{(\beta|T^*(\alpha))} = \overline{(T(\beta)|\alpha)} = (\alpha|T(\beta))$$

for all α and β.

Theorem 8.7: If T is a bounded linear operator on a Hilbert space H then T has a adjoint and this adjoint is bounded.

To see why this is true, first fix $\beta \in H$ and suppose that $\|T(\alpha)\| \leq M_T\|\alpha\|$ for all $\alpha \in H$. Then define a linear functional L on H by

$$L(\alpha) = (T(\alpha)|\beta).$$

L is bounded since

$$|L(\alpha)| = |(T(\alpha)|\beta)| \leq \|T(\alpha)\| \times \|\beta\| \leq M_T\|\alpha\| \times \|\beta\| = M_t\|\beta\| \times \|\alpha\|.$$

Therefore L is a bounded linear functional, so there is some γ so that

$$(T(\alpha)|\beta) = (\alpha|\gamma)$$

for all $\alpha \in H$. Set $S(\beta) = \gamma$. We can do this for every β, so we have a function $S : V \to V$ that satisfies

$$(T(\alpha)|\beta) = (\alpha|S(\beta))$$

for all α and β in H.

In order to show that S is the adjoint of T we have to show that S is linear. Suppose that $\beta_k \in H$ and x is a scalar. Then for all $\alpha \in H$:

$$\begin{aligned}(\alpha|xS(\beta_1) + S(\beta_2)) &= \overline{x}(\alpha|S(\beta_1)) + (\alpha|S(\beta_2)) \\ &= \overline{x}(T(\alpha)|\beta_1) + (T(\alpha)|\beta_2) \\ &= (T(\alpha)|x\beta_1 + \beta_2) \\ &= (\alpha|S(x\beta_1 + \beta_2))\end{aligned}$$

so $S(x\beta_1 + \beta_2) = xS(\beta_1) + S(\beta_2)$. Therefore T has an adjoint. Now we have to show that T^* is bounded.

Without loss of generality, suppose that $T^*(\alpha) \neq \vec{0}$. Then

$$\|T^*(\alpha)\|^2 = (T^*(\alpha)|T^*(\alpha)) = |(T(T^*(\alpha))|\alpha)| \leq (\|T(T^*(\alpha))\| \times \|\alpha\| \leq M_T\|T^*(\alpha)\| \times \|\alpha\|.$$

Now divide both sides by $\|T^*(\alpha)\|$ to get $\|T^*\alpha\| \leq M_T\|\alpha\|$.

Definition: If $T^* = T$ then we say that T is **self-adjoint**.

Theorem 8.8: Let H be a finite dimensional inner product space with ordered orthonormal basis $(\nu_1, \ldots, \nu_N)$ and let T be a linear operator on H. Then

$$\begin{aligned}[T](j,k) &= (T(\nu_j)|\nu_k) \\ [T^*](j,k) &= \overline{[T](k,j)}\end{aligned}$$

In particular, if T is self-adjoint then $[T] = [T]^*$, where all matrices of operators are computed with respect to the ordered orthonormal basis $(\nu_1, \ldots, \nu_N)$.

Theorem 8.9: Let V be an inner product space and let T and U be linear operators with adjoints T^* and U^*. Then $T + U$, cT and $U \circ T$ have adjoints $T^* + U^*$, $\overline{c}T^*$ and $T^* \circ U^*$ respectively.