Isomorphisms and Matrix Representations of Linear Transformations

All material from Chapter 3 of Linear Algebra by Hoffman and Kunze.

Definition: Suppose that $T \in L(V, W)$ and T is non-singular and onto W. Then we say that T is a **vector space isomorphism** and that V and W are **isomorphic** vector spaces.

Theorem 3.10: Any pair of vector spaces over F of the same dimension are isomorphic.

Definition: Suppose that $B = (\beta_1, \dots, \beta_n)$ is an ordered basis for V, that $G = (\gamma_1, \dots, \gamma_m)$ is an ordered basis for W and that $T \in L(V, W)$. Then $A \in F^{m \times n}$ whose k^{th} column is $[T(\beta_k)]_G$ is called **the matrix of T relative to the ordered bases B and G.**

Theorem 3.11: If B is an ordered basis for V, G is an ordered basis for W, $T \in L(V, W)$ and $A \in F^{m \times n}$ is the matrix of T relative to the ordered bases B and G then for each $\alpha \in V$,

$$[T(\alpha)]_G = A[\alpha]_B.$$

Furthermore, the map from $L(V, W)$ into $F^{m \times n}$ that associates to T its matrix relative to the ordered bases B and G is a vector space isomorphism.

Note: The preceeding theorem subsumes Theorem 3.12.

Theorem 3.13: Let V, W and Z be finite dimensional vector spaces over F with ordered bases B, G and H respectively. Let $T \in L(V, W)$ and $U \in L(W, Z)$. Let $[T]$ be the matrix of T relative to B and G, let $[U]$ be the matrix of U relative to G and H, and let $[U \circ T]$ be the matrix of $U \circ T$ with respect to B and H. Then

$$[U \circ T] = [U][T].$$

Notation: If $T \in L(V, V)$ and B is a finite basis for V then we will denote the matrix of T relative to B and B by $[T]_B$.

Theorem 3.14: Let V be a finite dimensional vector space with ordered bases B and G, and let $T \in L(V, V)$, and let $P \in F^{n \times n}$ satisfy

$$P[\alpha]_G = [\alpha]_B$$

for each $\alpha \in V$. (See Theorems 3.7 and 3.8.) Then

$$[T]_G = P^{-1}[T]_B P$$

In fact, if we define $S \in L(V, V)$ by $U(\beta_k) = \gamma_k$ for all $\beta_k \in B$ and $\gamma_k \in G$ then is the matrix of S relative to B and G.

Demonstration: It follows from Theorem 3.1 that S exists. It follows from Theorem 3.9 that S is invertible. It follows from Theorems 3.7 and 3.8 that P is the matrix matrix of S relative to B and G. Define $R \in L(V, V)$ by $R = S^{-1} \circ T \circ S$ and apply Theorem 3.13.

Definition: We say that A and $A' \in F^{n \times n}$ are **similar** if there is some invertible $P \in F^{n \times n}$ so that $P^{-1}AP = A'$.

Proposition: If A and A' are similar, they have the same determinant.

Definition: If V is a finite dimensional vector space and $T \in L(V, V)$, then **determinant of** T is the determinant of $[T]_B$ for any ordered basis B of V.