Linear Transformations

All material from Chapter 3 of Linear Algebra by Hoffman and Kunze.

Definition: If V and W are vector spaces and $T : V \to W$, then we say that T is a **linear transformation from** V **into** W if for any pair of vectors α and β in V and any scalar $c \in F$ we have
$$T(c\alpha + \beta) = cT(\alpha) + T(\beta).$$

Theorem 3.1: Suppose that

- V and W are vector spaces over F,
- $B := \{\beta_a, a \in A\}$ is a basis for V,
- $S \subset W$
- $G : B \to S$.

Then there is exactly one linear transformation $T : V \to W$ so that $T(\beta_a) = G(\beta_a)$ for all $a \in A$.

Demonstration: We have to show there is at least one such linear transformation, and not more than one such linear transformation.

First we show there is such a linear transformation. Let $\alpha \in V$. Then there is precisely one set of coefficients in F so that
$$\alpha = \sum_{a \in A} b_a \beta_a.$$

Since B is a basis, this is a finite sum for any particular α since all but finitely many of the b_a must be 0. Define $T : V \to W$ by
$$T(\alpha) = \sum_{a \in A} b_a G(\beta_a).$$

We have to show that T is a linear transformation. Choose σ and $\phi \in V$, and $c \in F$. We know that
$$\begin{aligned} \sigma &= \sum_{a \in A} s_a \beta_a \\ \phi &= \sum_{a \in A} f_a \beta_a \\ c\sigma + \phi &= \sum_{a \in A} (cs_a + f_a)\beta_a, \end{aligned}$$

these linear combinations are unique and the sums are finite. Therefore
$$T(c\sigma + \phi) = \sum_{a \in A} (cs_a + f_a)G(\beta_a) = c\sum_{a \in A} s_a G(\beta_a) + \sum_{a \in A} f_a G(\beta_a) = cT(\sigma) + T(\phi).$$

Now, suppose that S is another linear transformation for which $S(\beta_a) = G(\beta_a)$ for each $a \in A$. To show that $S = T$ we have to show that $S(\alpha) - T(\alpha) = \vec{0}$ for every $\alpha \in V$. Observe that
$$S(\alpha) - T(\alpha) = \left(\sum_{a \in A} b_a S(\beta_a)\right) - \left(\sum_{a \in A} b_a T(\beta_a)\right) = \left(\sum_{a \in A} b_a G(\beta_a)\right) - \left(\sum_{a \in A} b_a G(\beta_a)\right) = \vec{0}$$

Notation: If V and W are vector spaces over F, then $L(V, W)$ denote the set of all linear transformations from V into W.

Definition: Let $T \in L(V, W)$. The **null space** of T, denoted by $N(T)$, is $N(T) = \{\alpha \in V : T(\alpha) = \vec{0}\}$.

Proposition: If $T \in L(V,W)$, then $N(T)$ is a subspace of V, and the range of T is a subspace of W. We will denote the range of T by $R(T)$.

Defintion: If $N(T)$ is finite dimensional then its dimension is called the bf nullity of T. If $R(T)$ is finite dimensional its dimension is called the **rank** of T. We will denote the nullity by $n(T)$ and the rank by $r(T)$.

Theorem 3.2: If V is finite dimensional and $T \in L(V,W)$ then

$$r(T) + n(T) = \dim(V).$$

Theorem 3.4: $L(V,W)$ is a vector space over the field F common to V and W with the usual defintions of addition and scalar multiplication of functions.

Note: We have already noted that the set of all functions from V to W is a vector space over F, so it is sufficient to show that $L(V,W)$ is not empty and is closed under addition of functions and scalar multiplication of functions.

Theorem 3.5: If V and W are finite dimensional then

$$\dim(L(V,W)) = \dim(V) \times \dim(W).$$

Theorem 3.6: If $U \in L(W,Z)$ and $T \in L(V,W)$ then $U \circ T \in L(V,Z)$. ($U \circ T$ is the composition of U and T.)

Definition: If $T \in L(V,V)$ then T is called a **linear operator on** V. $I \in L(V,V)$ is the identity function: $I(\alpha) = \alpha$ for all $\alpha \in V$.

Lemma: Let U, T and S be linear operators on V, a vector space over F. Let $c \in F$. Then

1. $I \circ U = U \circ I = U$;
2. $U \circ (S + T) = (U \circ S) + (U \circ T)$;
3. $(S + T) \circ U = (S \circ U) + (T \circ U)$;
4. $c(U \circ T) = (cU) \circ T = U \circ (cT)$

Definition: If $T \in L(V,W)$ and there are $U_i : W \to V$ such that $U_1 \circ T = T \circ U_2 = I$ then $U_1 = U_2$ and we say that T is **invertible** and U_1 is called the **inverse of** T and will be denoted by T^{-1}.

Theorem 3.7: If $T \in L(V,W)$ and T is invertible, then $T^{-1} \in L(W,V)$.

Definition: If $T \in L(V,W)$ and $T(\alpha) = \vec{0}_W$ implies $\alpha = \vec{0}_V$ then we say that T is **non-singular**. Since T is linear, non-singular is equivalent to **one-to-one**.

Theorem 3.8: Suppose that $T \in L(V,W)$. T is non-singular if and only if T carries each linearly independent subset of V to a linearly independent subset of W.

Theorem 3.9: Let V and W be finite dimensional vectors spaces of equal dimension, and suppose that $T \in L(V,W)$. Then the following statements are equivalent.

1. T is invertible.
2. T is non-singular.
3. $n(T) = 0$.
4. The range of T is W.
5. $r(T) = \dim(W)$.
6. If $\{\beta_1, \ldots, \beta_n\}$ is a basis for V then $\{T(\beta_1), \ldots, T(\beta_n)\}$ is a basis for W.
7. There is some basis $\{\beta_1, \ldots, \beta_n\}$ for V such that $\{T(\beta_1), \ldots, T(\beta_n)\}$ is a basis for W.

Definition: A set G closed under an operation $\cdot$ is called a group if

1. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ for all $g_i \in G$.
2. There is some $\mathrm{e} \in G$ so that $\mathrm{e} \cdot g = g \cdot \mathrm{e} = g$ for all $g \in G$.
3. For each $g \in G$ there is some $g' \in G$ so that $g \cdot g' = g' \cdot g = \mathrm{e}$.