

Linear Functionals

All material from Chapter 3 and 8 of Linear Algebra by Hoffman and Kunze.

Definition: Let V be a vector space over a field F . Since F is a vector space over F we may consider the set $L(V, F)$. Elements of $L(V, F)$ are called **linear functionals**. $L(V, F)$ is called the **dual space** of V and is denoted by V^* . We know that V^* is a vector space over F and that if V is finite dimensional then $\dim(V) = \dim(V^*)$. This tells us that V and V^* are isomorphic if $\dim(V) < \infty$. Shortly we will find a *nice* isomorphism.

Three important examples: These are probably the three most important examples of where linear functionals come from.

1. Suppose that F is a field, and S is a non-empty set. We know that $V = \{f : S \rightarrow V\}$ is a vector space. For each $s \in S$ we may define a linear functional L_s by

$$L_s(f) = f(s).$$

These are called the **evaluation linear functionals**.

2. Linear functionals on inner product spaces. Suppose that V is an inner product space. Then each $\beta \in V$ defines a linear functional L_β given by

$$L_\beta(\alpha) = (\alpha|\beta).$$

For example, if $F = C$, the complex numbers, $(\cdot|\cdot)$ is the standard inner product on C^3 , $\beta = (2 + i, 3 - i, i)$, and $\alpha = (x, y, z)$ then

$$L_\beta(\alpha) = ((x, y, z)|(2 + i, 3 - i, i)) = (2 - i)x + (3 + i)y - iz.$$

For another example, suppose that V is the real valued continuous functions with domain $[-1, 1]$, the field is the real numbers, the inner product is the standard inner product, and β is the constant function 1. Then L_1 is given by

$$L_1(f) = \int_{-1}^1 f(x) dx.$$

3. Determinants. Suppose that F is a field, $n > 1$ is an integer and $\beta_2, \dots, \beta_n \in F^n$ are given to us. Then for each $\alpha \in F^n$ we may define a matrix whose first row is α , second row is β_2 and so on. Then since the determinant function is n -linear, the function $L : F^n \rightarrow F$ defined by

$$L(\alpha) = \det(\alpha, \beta_2, \dots, \beta_n)$$

is a linear functional. For example, if $n = 2$, $\beta_2 = (2, 5)$, and $\alpha = (x, y)$ then

$$L(\alpha) = L((x, y)) = \det \left(\begin{bmatrix} x & y \\ 2 & 5 \end{bmatrix} \right) = 5x - 2y$$

Another example. If $n = 3$, $\beta_2 = (3, 4, 7)$, $\beta_3 = (2, 1, 4)$ and $\alpha = (x, y, z)$ then

$$L(\alpha) = L((x, y, z)) = \det \left(\begin{bmatrix} x & y & z \\ 3 & 4 & 7 \\ 2 & 1 & 4 \end{bmatrix} \right) = 9x + 2y - 5z,$$

We shall see that this last example is related to cross product.

Definition: Suppose that B is a basis for V . Then for each $\beta \in B$ there is a unique (see Theorem 3.1) linear functional $\beta^* \in V^*$ such that $\beta^*(\beta) = 1$ and $\beta^*(\alpha) = 0$ if $\alpha \in B$ and $\alpha \neq \beta$. β^* is called the **dual of β** .

Lemma: If B is a linearly independent set in V then the set $\{\beta^* \in V^*\}$ is a linearly independent subset of V^* .

Demonstration: Let $Z \in V^*$ be the zero element of V^* , that is $Z(\alpha) = 0$ for every $\alpha \in V$. Suppose that

$$c_1\beta_1^* + \cdots + c_n\beta_n^* = Z.$$

Then

$$\begin{aligned} c_1\beta_1^*(\beta_1) + \cdots + c_n\beta_n^*(\beta_1) &= Z(\beta_1) \\ (c_1 \times 1) + (c_2 \times 0) \cdots + (c_n \times 0) &= 0 \end{aligned}$$

so $c_1 = 0$. In the same way, $c_2 = \cdots = c_n = 0$.

Definition: If the collection $\{L_\beta\}$ is a basis of V^* it is called the **dual basis**.

Theorem 3.15 Let V be a finite dimensional vector space over the field F . Let $(\beta_1, \dots, \beta_n)$ be an ordered basis for V , and put $\beta_k = L_{\beta_k}$. Let I denote the $n \times n$ identity matrix in $F^{n \times n}$. Then $(\beta_1, \dots, \beta_n)$ is the unique ordered basis in of V^* with the following properties:

1. $\beta_k^*(\beta_j) = I_{k,j}$ for $k, j \in \{1, \dots, n\}$;
2. If $\beta^* \in V^*$ then

$$\beta^* = \sum_{k=1}^n \beta^*(\beta_k)\beta_k^*$$

3. If $\alpha \in V$ then

$$\alpha = \sum_{k=1}^n \beta^*(\alpha)\beta_k.$$

Corollary: Let V is a finite dimensional vector space over F with ordered basis

$$B = \{\beta_1, \dots, \beta_n\}.$$

Let

$$B^* = \{\beta_1^*, \dots, \beta_n^*\}$$

be the corresponding dual basis for V^* . Then for each $\alpha \in V$ and $f \in V^*$

$$[f]_{B^*} = \begin{bmatrix} f(\beta_1) \\ \vdots \\ f(\beta_n) \end{bmatrix}$$

$$[\alpha]_B = \begin{bmatrix} \beta_1^*(\alpha) \\ \vdots \\ \beta_n^*(\alpha) \end{bmatrix}.$$

$$[f] = [f(\beta_1) \quad \cdots \quad f(\beta_n)]$$

$$f(\alpha) = \sum_{k=1}^n \beta_k^*(\alpha)f(\beta_k)$$

Riesz Representation Theorem: Let S be a finite set of n elements, let F be a field, and let $V = \{f : S \rightarrow F\}$. Let $f \in V^*$. Then there is a unique $\phi \in V$ so that for all $\alpha \in F^n$

$$f(\alpha) = \sum_{s \in S} \alpha(s)\phi(s)$$

Corollary: Let V be a finite dimensional vector space over the field F with ordered basis $(\beta_1, \dots, \beta_n)$. Define $T : V \rightarrow V^*$ by

$$T(\alpha) = \sum_{k=1}^n \beta_k^*(\alpha) \beta_k^*$$

and $S : V^* \rightarrow V$ by

$$S(f) = \sum_{k=1}^n f(\beta_k) \beta_k.$$

Then T is the isomorphism between V and V^* that carries each β_k to its dual element β_k^* and S is its inverse.

Interpolating Polynomials: Let F be a field, let $S \subset F$ be non-empty, and let $V = \{f : s \rightarrow F\}$. Let s_1, s_2, \dots, s_n be $n \geq 1$ distinct elements in S and define the polynomial functions

$$\begin{aligned} p(x) &= (x - s_1)(x - s_2) \cdots (x - s_n) \\ p_1(x) &= p(x)/(x - s_1) \\ &\vdots \\ p_n(x) &= p(x)/(x - s_n) \end{aligned}$$

(We adopt the convention that the polynomial division on the right hand sides is done formally to avoid confusion about division by 0.) Form new polynomial functions q_k by $q_k(x) = p_k(x)/p_k(s_k)$ so that $q_k(s_j) = I_{k,j}$.

Then the polynomials q_k are a basis for the space of polynomials of degree less than n over F and the evaluation functionals $L_k(f) = f(s_k)$ are their duals.

Definition: Suppose that $B = (\beta_1, \dots, \beta_n)$ is an ordered basis of R^n . We say that B is **righthanded** if $\det(\beta_1, \dots, \beta_n) > 0$. We say that B is **lefthanded** if $\det(\beta_1, \dots, \beta_n) < 0$.

Cross Product: Let R denote the real numbers and let $V = R^3$. V is the same as $\{f : \{1, 2, 3\} \rightarrow R\}$. Let $(\cdot|\cdot)$ be the standard inner product on V . Fix β and $\gamma \in V$ and define $L \in V^*$ by $L(\alpha) = \det(\alpha, \beta, \gamma)$ as in the examples above. Then according to the Riesz Representation Theorem there is some $\phi \in V$ so that $L(\alpha) = (\alpha|\phi)$. From the properties of determinants we know

$$\begin{aligned} (\beta|\phi) &= \det(\beta, \beta, \gamma) = 0 \\ (\gamma|\phi) &= \det(\gamma, \beta, \gamma) = 0 \end{aligned}$$

Furthermore, $\phi = (0, 0, 0)$ if and only if β and γ are linearly independent, for

$$0 \leq (\phi|\phi) = \det(\phi, \beta, \gamma).$$

If β and γ are linearly dependent, then $\det(\phi, \beta, \gamma) = 0$, so $\phi = (0, 0, 0)$. If β and γ are linearly independent, then there is some $\alpha \in V$ so that (α, β, γ) are linearly independent, and, therefore, are the rows of an invertible matrix. Then

$$(\alpha|\phi) = \det(\alpha, \beta, \gamma) \neq 0$$

so $\phi \neq \vec{0}$ and

$$\det(\phi, \beta, \gamma) = (\phi|\phi) > 0.$$

The vector ϕ is called the **cross product** of β and γ and is denoted by $\beta \times \gamma$.

If β and γ are linearly independent then $(\beta \times \gamma, \beta, \gamma)$ is a righthanded basis for R^3 .

To see more properties of cross product, go to

<http://www.uwm.edu/~ericskey/535material/crossprod.pdf>

Definition: Suppose that V is an inner product space and $f \in V^*$. We say that f is **bounded** if there is a real number M so that

$$|f(\alpha)| \leq M\|\alpha\|.$$

An inner product space H is called a **Hilbert space** (named after the mathematician David Hilbert) if every for each bounded linear functional f there is some $\phi \in H$ so that

$$f(\alpha) = (\alpha|\phi)$$

for every $\alpha \in H$. Note that ϕ must be unique for if $(\alpha|\phi) = (\alpha|\phi')$ for all α then $\phi = \phi'$.

Theorem: If V is an inner product space, $f \in V^*$ and $f(\alpha) = (\alpha|\phi)$ for all $\alpha \in V$ then $V = N(f) \oplus N(f)^\perp$.

Demonstration: The theorem is clearly true if $f(\alpha) = 0$ for all α . If not, then choose ϕ so that $f(\alpha) = (\alpha|\phi)$. We know that $\phi \neq \vec{0}$. If $\alpha \in V$ then

$$\alpha = \text{proj}_\phi(\alpha) + (\alpha - \text{proj}_\phi(\alpha)).$$

Since we have

$$\begin{aligned} f(\alpha - \text{proj}_\phi(\alpha)) &= ((\alpha - \text{proj}_\phi(\alpha))|\phi) \\ &= 0 \end{aligned}$$

we see that

$$\begin{aligned} \alpha - \text{proj}_\phi(\alpha) &\in N(f) \\ \text{proj}_\phi(\alpha) &\in N(f)^\perp \end{aligned}$$

Since we know that for every subspace W , $W \cap W^\perp = \{\vec{0}\}$ we have proven our claim.

Theorem 8.6: Every finite dimensional inner product space is a Hilbert space.

Demonstration: Let (ν_1, \dots, ν_n) be an orthonormal basis of V and put

$$\phi = \sum_{k=1}^n \overline{f(\nu_k)} \nu_k$$

Then it is easy to check that $f(\alpha) = (\alpha|\phi)$ for every $\alpha \in H$.

Definition: If V is a vector space, and W and W' are subspaces of V , we say that W is a **hyperspace** or **hyperplane** if

- $V = W \oplus W'$;
- $\dim(W') = 1$.

For this reason a hyperspace is also called a **co-dimension one** since the dimension of the “complementary” space W' is one. If the dimension of V is n , then every subspace of dimension $n - 1$ is a hyperspace. In an inner product space V , if $\alpha \neq \vec{0}$ and $V = \text{span}\alpha \oplus \{\alpha\}^\perp$ then $\{\alpha\}^\perp$ is a hyperspace. In a Hilbert space, the null space of every bounded linear functional that is not identically 0 is a hyperspace.

Definition: If V is a vector space over a field F and S is a subspace of V , the **annihilator** of S is the subspace of V^* , call it S^0 , with the property that $f \in S^0$ if and only if $f(\sigma) = 0$ for each $\sigma \in S$. Observe that $\dim(V^0) = 0$ and $\{\vec{0}\}^0 = V^*$.

Theorem 3.16: If V is a finite dimensional vector space and W is a subspace of V then

$$\dim(W) + \dim(W^0) = \dim(V).$$

Corollary: If W is a k dimensional subspace of an n dimensional vector space V then W is the intersection of $(n - k)$ hyperspaces in V .

Corollary: If W_1 and W_2 are subspace of a finite dimensional vector space V then $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.