Determinants: Lecture 02

All material from Chapter 5 of Linear Algebra by Hoffman and Kunze.

Definition : A **permutation of degree** n is a one-to-one function from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$.

We can represent a permutation of degree n as an n-tuple of elements of $\{1, 2, \ldots, n\}$, $(k_1, k_2, \ldots, k_n)$ where $k_i \neq k_j$ when $i \neq j$. If $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ then we may list out σ as an n-tuple: $(\sigma(1), \ldots, \sigma(n))$.

Observation : If $\epsilon_1, \ldots, \epsilon_n$ are the standard basis vectors in $K^{1\times n}$ and D is an alternating n-linear function on $K^{n\times n}$ then

$$D(A) = \sum_{\sigma\in\Sigma} \left(\prod_{i=1}^{n} A(i, \sigma(i)) \right) D\left(\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(n)}\right)$$

where Σ is the set of all permutations of degree n.

Observation : Note that

$$D\left(\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(n)}\right) = \pm 1.$$

Definition: A permutation σ is called a **interchange** or **transposition** if $\sigma(k) = k$ for all but two values of k. If σ can be written as the composition of an even number of transpositions then we say that σ is **even**, and we write $\operatorname{sgn}(\sigma) = 1$. If it can be written as an odd number of transpositions we say that it is **odd** and we write $\operatorname{sgn}(\sigma) = -1$.

Note : We have to show that the function $\operatorname{sgn} : \Sigma \to \{-1, 1\}$ is well-defined.

Theorem 2 : Let K be a commutative ring with identity and let n be a positive integer. There is precisely one determinant function on $K^{n\times n}$, call it $\det_n$, and it can be computed using the formula

$$\det_n(A) = \sum_{\sigma\in\Sigma} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{n} A(i, \sigma(i)) \right).$$

Theorem 3: Let K be a commutative ring with identity and let $A \in K^{n\times n}$ and $B \in K^{n\times n}$. Then

$$\det_n(AB) = \det_n(A) \det_n(B).$$

Corollary: $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$.

Properties of determinants: In what follows det is a generic determinant function.

1. If A^t denotes the transpose of A, then $\det(A) = \det(A^t)$.
2. If B is obtained from A by adding a multiple of one row of A to another row of A then $\det(A) = \det(B)$.
3. Suppose m and n are positive integers and
   - $A \in K^{r\times r}$;
   - $B \in K^{r\times s}$;
   - $D \in K^{s\times s}$;
   - $[0]$ is the zero matrix in $K^{s\times r}$.

   If $M \in K^{(r+s)\times(rs)}$ is given by

   $$M = \begin{bmatrix} A & B \\ [0] & D \end{bmatrix}$$

   then $\det_{r+s}(M) = \det_r(A) \det_s(D)$.

Definition: The quantity $C_{i,j} := (-1)^{i+j} \det(A(i|j))$ is called the $\boldsymbol{i}$**,** $\boldsymbol{j}$ **cofactor** of A.

Definition: The matrix $\mathrm{adj}(A)$ defined by $(\mathrm{adj}(A))_{i,j} = C_{j,i}$ is called the **classical adjoint** of A.

Theorem 4: We have

$$\mathrm{adj}(A)A = A\mathrm{adj}(A) = \det(A)I$$

Therefore A is invertible if and only if $\det(A)$ is invertible.

Cramer's Rule: If $AX = Y$ where $A \in K^{n\times n}$, $X \in K^{n\times 1}$ and $Y \in K^{n\times 1}$ then

$$\det(A)X_k = \det(B_k)$$

where $B_k(i,j) = A(i,j)$ if $j \neq k$ and $B_k(i,k) = Y_i$, that is we replace column k of A with Y.