Determinants: Lecture 01

All material from Chapter 5 of Linear Algebra by Hoffman and Kunze.

Definition: A **ring** is a set K, together with two operations $(x, y) \to x + y$ (addition) and $(x, y) \to xy$ (multiplication) satisfying

1. Addition is a commutative operation;
2. Addition is an associative operation;
3. There is an element $0 \in K$ so that $x + 0 = x$ for all $x \in K$;
4. For each $x \in K$ there is some $-x \in K$ so that $-x + x = 0$;
5. Multiplication is associative;
6. Multiplication distributes over addition:
   - $(x + y)z = xz + yz$
   - $z(x + y) = zx + zy$

If the multiplication is commutative we say the ring is **commutative**. If there is some element $1 \in K$ so that $1x = x1 = x$ for all $x \in K$ we say that K is a **ring with identity** and 1 is called the **identity** for K.

Definition: Let K be a commutative ring with identity, n a positive integer, and let D be a function which assigns each $n \times n$ matrix A over K a scalar $D(A) \in K$. We say that D is n**-linear** if for each $i \in \{1, \dots, n\}$, D is a linear function of the i^{th} row when the other $n - 1$ rows are held fixed.

Lemma: A linear combination of n-linear functions is n-linear.

Definition: Let D be an n-linear function. We say D is **alternating** if the following two conditions are satisfied:

- $D(A) = 0$ whenever two rows of A are equal;
- If B is the matrix obtained from A by interchanging two rows of A then $D(B)+D(A) = 0$.

Definition: Let K be a commutative ring with identity, and let n be a positive integer. Suppose D is a function from the $n \times n$ matrices over K. We say that D is a **determinant function** if

- D is n-linear;
- D is alternating;
- D sends the identity matrix I to 1: $D(I) = 1$.

Lemma: Let D be a 2-linear function with the property that $D(A) = 0$ for all 2×2 matrices $A \in K^{2\times 2}$ with equal rows. Then D is alternating.

Lemma: Let D be an n-linear function on $K^{n\times n}$ Suppose that for any matrix $A \in K^{n\times n}$ that has two adjacent rows equal, $D(A) = 0$. Then D is alternating.

Definition: If $n > 1$ and $A \in K^{n\times n}$, we let $A(i|j)$ denote the element of $K^{(n-1)\times(n-1)}$ obtained by deleting the i^{th} row of A and the j^{th} column of A. If D is an $(n - 1)$-linear function and $A \in K^{n\times n}$ we put $D_{i,j}(A) = D[A(i|j)]$.

Theorem 1 *Let $n > 1$ and let D be an alternating $(n - 1)$-linear function on $K^{(n-1)\times(n-1)}$. For each $j \in \{1, \dots, n\}$ the function E_j defined by*

$$E_j(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A) \qquad (1)$$

is an alternating n-linear function on $K^{n\times n}$. If D is a determinant function then so is each E_j.

Corollary: Let K be a commutative ring with identity and let n be a positive integer. There exists at least one determinant function on $K^{n\times n}$.