A Coordinate-free Approach to Cross-Product

In what follows **boldface** indicates vector quantities in three dimensions, other variables are scalars.

The ordered triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is said to be right-handed if

$$\det\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix}\right) > 0.$$

It is called **left-handed** if this determinant is negative. It is easy to check from the alternating property of determinants the affect that permuting the elements of a triple of vectors has on its "handedness", and that standard basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is right-handed.

1 Cross Product

Given $\mathbf{B}$ and $\mathbf{C}$ let us define $L : R^3 \to R$ by

$$L(\mathbf{X}) = \det\left(\begin{bmatrix} \mathbf{X} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix}\right).$$

The multi-linear property of determinants tells us that L is a **linear functional**, and so there is a unique vector $\mathbf{B} \times \mathbf{C}$, called the **cross product of** B **and** C, with the property that for every $\mathbf{X}$,

$$\det\left(\begin{bmatrix} \mathbf{X} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix}\right) = \mathbf{X} \cdot (\mathbf{B} \times \mathbf{C}). \tag{1}$$

We will now proceed to show how all of the properties of cross product may be derived from (1) and the properties of determinants. Before we begin, we recall two properties of the **dot product**:

- If for all $\mathbf{X}$ we have $\mathbf{X} \cdot \mathbf{A} = \mathbf{X} \cdot \mathbf{B}$ then $\mathbf{A} = \mathbf{B}$, since by choosing $\mathbf{X} = \mathbf{A} - \mathbf{B}$ and subtracting we have
$$0 = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = ||(\mathbf{A} - \mathbf{B})||^2$$
which gives $\mathbf{A} - \mathbf{B} = \mathbf{0}$, that is $\mathbf{A} = \mathbf{B}$.
- $(\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})^2 = 0$ if and only if $\mathbf{A}$ and $\mathbf{B}$ are linearly dependent. To see see, if $\mathbf{A}$ and $\mathbf{B}$ are linearly dependent, then one is a scalar multiple of the other, and it is easy to see that the formula holds. If not, then neither vector is the zero vector, and there is a non-zero vector $\mathbf{C}$ and a non-zero scalar b so that $\mathbf{A} \cdot \mathbf{C} = 0$ and $\mathbf{B} = b\mathbf{A} + \mathbf{C}$. Then $\mathbf{B} \cdot \mathbf{B} = b^2(\mathbf{A} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{C}$ and $\mathbf{A} \cdot \mathbf{B} = b(\mathbf{A} \cdot \mathbf{A})$ from which we easily see by substitution that $(\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})^2 \neq 0$.

  This may be restated by saying that vectors are dependent if and only if
$$||\mathbf{A}|| \cdot ||\mathbf{B}|| = |\mathbf{A} \cdot \mathbf{B}|$$
or that the cosine of the angle between them is 1 or -1 (if neither vector is the zero vector).

Now we may derive various properties of cross products.

- $\mathbf{B} \times \mathbf{C}$ is orthogonal to $\mathbf{B}$ and to $\mathbf{C}$.
$$\mathbf{B} \cdot (\mathbf{B} \times \mathbf{C}) = \det\left(\begin{bmatrix} \mathbf{B} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix}\right) = 0$$
since we have the determinant of a matrix with linearly dependent rows. The same argument works for $\mathbf{C}$.

- $\mathbf{B}$ and $\mathbf{C}$ are linearly dependent if and only if $\mathbf{B} \times \mathbf{C} = \mathbf{0}$. To see why, observe that

$$||\mathbf{B} \times \mathbf{C}||^2 = (\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{B} \times \mathbf{C}) = \det \left(\begin{bmatrix} \mathbf{B} \times \mathbf{C} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix} \right).$$

  If $\mathbf{B}$ and $\mathbf{C}$ are dependent then this determinant is 0, and so $\mathbf{B} \times \mathbf{C}$, having length 0, must be the zero vector. If $\mathbf{B}$ and $\mathbf{C}$ are linearly independent, then there is some $\mathbf{X}$ so that $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{X}$ are linearly independent so that $\mathbf{X} \cdot (\mathbf{B} \times \mathbf{C}) \neq 0$. Hence $\mathbf{B} \times \mathbf{C} \neq \mathbf{0}$.

- If $\mathbf{B}$ and $\mathbf{C}$ are linearly independent then $(\mathbf{B} \times \mathbf{C}, \mathbf{B}, \mathbf{C})$ is a right-handed triple of vectors. Just apply the same computation in the previous item to see that this assertion is true as long as $\mathbf{B} \times \mathbf{C} \neq \mathbf{0}$, which is true since $\mathbf{B}$ and $\mathbf{C}$ are not dependent.

- The order of the factors in the crossproduct matters, since

$$\mathbf{X} \cdot (\mathbf{B} \times \mathbf{C}) = \det \left(\begin{bmatrix} \mathbf{X} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix} \right) = -\det \left(\begin{bmatrix} \mathbf{X} \\ \mathbf{C} \\ \mathbf{B} \end{bmatrix} \right) = -\mathbf{X} \cdot (\mathbf{C} \times \mathbf{B}) = \mathbf{X} \cdot (-\mathbf{C} \times \mathbf{B})$$

  Therefore

$$\mathbf{B} \times \mathbf{C} = -\mathbf{C} \times \mathbf{B}.$$

  This is called the **anti-symmetric property**.

- The cross product is linear in its factors. For any $\mathbf{X}$ we have

$$\begin{aligned} \mathbf{X} \cdot ((x\mathbf{B} + \mathbf{D}) \times \mathbf{C}) &= \det \left(\begin{bmatrix} \mathbf{X} \\ x\mathbf{B} + \mathbf{D} \\ \mathbf{C} \end{bmatrix} \right) \\ &= x \det \left(\begin{bmatrix} \mathbf{X} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix} \right) + \det \left(\begin{bmatrix} \mathbf{X} \\ \mathbf{D} \\ \mathbf{C} \end{bmatrix} \right) \\ &= x\mathbf{X} \cdot (\mathbf{B} \times \mathbf{C}) + \mathbf{X} \cdot (\mathbf{D} \times \mathbf{C}) \\ &= \mathbf{X}(x(\mathbf{B} \times \mathbf{C}) + \mathbf{D} \times \mathbf{C}) \end{aligned}$$

  so

$$(x\mathbf{B} + \mathbf{D}) \times \mathbf{C} = x(\mathbf{B} \times \mathbf{C}) + \mathbf{D} \times \mathbf{C}.$$

  This shows that the cross-product is linear in the first factor. To show linearity in the second factor, one can either repeat this argument or use the anti-symmetry property to shift the second factor to the first and back again.

- The length of the cross-product. We begin with the following matrix equation:

$$\begin{bmatrix} \mathbf{A} \times \mathbf{B} \\ \mathbf{A} \\ \mathbf{B} \end{bmatrix} [\mathbf{A} \times \mathbf{B} \; \mathbf{C} \; \mathbf{D}] = \begin{bmatrix} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) & (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} & (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D} \\ 0 & \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ 0 & \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{bmatrix}.$$

  Compute the determinant of both sides of this equation, using that the determinant of a product is the product of the determinants, that a matrix and its transpose have the same determinant, and that the determinant of the first factor is $||\mathbf{A} \times \mathbf{B}||^2$, and the determinant of the second factor is $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$ to get

$$||\mathbf{A} \times \mathbf{B}||^2 (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = ||\mathbf{A} \times \mathbf{B}||^2 \det \left(\begin{bmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{bmatrix} \right). \quad (2)$$

  We know that $||\mathbf{A} \times \mathbf{B}||^2 = 0$ if and only if $\mathbf{A}$ and $\mathbf{B}$ are co-linear, and if $\mathbf{A}$ and $\mathbf{B}$ are colinear then it is easy to check that

$$\det \left(\begin{bmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{bmatrix} \right) = 0.$$

Therefore, (2) reduces to

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = \det\left(\begin{bmatrix} \mathbf{A}\cdot\mathbf{C} & \mathbf{A}\cdot\mathbf{D} \\ \mathbf{B}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{D}\end{bmatrix}\right) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}). \tag{3}$$

If we take $\mathbf{C} = \mathbf{A}$ and $\mathbf{D} = \mathbf{B}$ we get

$$||\mathbf{A}\times\mathbf{B}||^2 = (\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})^2.$$

If we use the geometric definition of dot product in terms of the cosine of the angle between the vectors (valid only for a pair of non-zero vectors) we get

$$||\mathbf{A}\times\mathbf{B}||^2 = ||\mathbf{A}||^2||\mathbf{B}||^2 - ||\mathbf{A}||^2||\mathbf{B}||^2\cos^2(\theta) = ||\mathbf{A}||^2||\mathbf{B}||^2\sin^2(\theta)$$

telling us that the length of the cross product is the product of the length of the vectors times the sine of the angle between them, which is the area of the parallelogram spanned by these vectors. This, in turn, shows that

$$|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$$

gives the volume of the parallelopiped generated by the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

- A **triple product identity**. The quantity $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ is sometimes called the triple product of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. It properties are easily deduced by remembering that

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \det\left(\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{bmatrix}\right)$$

and the appropriate properties of determinants. For example,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \det\left(\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{bmatrix}\right) = -\det\left(\begin{bmatrix}\mathbf{B}\\ \mathbf{A}\\ \mathbf{C}\end{bmatrix}\right) = \det\left(\begin{bmatrix}\mathbf{C}\\ \mathbf{A}\\ \mathbf{B}\end{bmatrix}\right) = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

- Cross product is not associative. This is easy to see by picking two non-zero orthogonal vectors $\mathbf{B}$ and $\mathbf{C}$ and comparing:

$$\begin{aligned}(\mathbf{B}\times\mathbf{B})\times\mathbf{C} &= \mathbf{0}\\ \mathbf{B}\times(\mathbf{B}\times\mathbf{C}) &\neq \mathbf{0}\end{aligned}$$

where the inequality follows by computing the length of the double cross product. Suppose then that $\mathbf{B}$ and $\mathbf{C}$ are linearly independent vectors. We consider $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. We know that $\mathbf{B}\times CC \neq \vec{0}$ so there are constants x, y and z so that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = x\mathbf{B} + y\mathbf{C} + z(\mathbf{B}\times\mathbf{C}).$$

We proceed now to find x and y and z by dotting the preceeding equation with $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{B}\times\mathbf{C}$ in turn:

$$\mathbf{B}\cdot(\mathbf{A}\times(\mathbf{B}\times\mathbf{C})) = x(\mathbf{B}\cdot\mathbf{B}) + y(\mathbf{B}\cdot\mathbf{C}) \tag{4}$$

$$\mathbf{C}\cdot(\mathbf{A}\times(\mathbf{B}\times\mathbf{C})) = x(\mathbf{B}\cdot\mathbf{C}) + y(\mathbf{C}\cdot\mathbf{C}) \tag{5}$$

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{A}\times(\mathbf{B}\times\mathbf{C})) = z||\mathbf{B}\times\mathbf{C}||^2 \tag{6}$$

where we take advantage that a the cross product of two vectors is orthogonal to each of the vectors. Next, from the properties of triple products above, we see that (4), (5) and (6) reduce to

$$(\mathbf{B}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{C})) = x(\mathbf{B}\cdot\mathbf{B}) + y(\mathbf{B}\cdot\mathbf{C}) \tag{7}$$

$$(\mathbf{C}\times\mathbf{A})\cdot(\mathbf{B}\times\mathbf{C})) = x(\mathbf{B}\cdot\mathbf{C}) + y(\mathbf{C}\cdot\mathbf{C}) \tag{8}$$

$$0 = z||\mathbf{B}\times\mathbf{C}||^2, \tag{9}$$

so that we immediately see $z = 0$. As for (7) and (8) we now apply (3) to the left-hand side and write the results in matrix form:

$$\begin{bmatrix} (\mathbf{B}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{B}) \\ (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{C}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{B}) \end{bmatrix} = \begin{bmatrix} \mathbf{B}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{C} \\ \mathbf{B}\cdot\mathbf{C} & \mathbf{C}\cdot\mathbf{C} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{10}$$

where the 2×2 matrix is invertible, since (3) shows its determinant is the square of the length of $\mathbf{B} \times \mathbf{C}$. Moreover, the vector on the lefthand side of (10) may be written as

$$\begin{bmatrix} (\mathbf{B}\cdot\mathbf{B})(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{B}) \\ (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{C}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{B}) \end{bmatrix} = \begin{bmatrix} \mathbf{B}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{C} \\ \mathbf{B}\cdot\mathbf{C} & \mathbf{C}\cdot\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{A}\cdot\mathbf{C} \\ -\mathbf{A}\cdot\mathbf{B} \end{bmatrix} \tag{11}$$

so by comparing (10) and (11) we see that $x = \mathbf{A}\cdot\mathbf{C}$ and $y = -\mathbf{A}\cdot\mathbf{B}$, yielding

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}, \tag{12}$$

which is easily seen to be true even when $\mathbf{B}$ and $\mathbf{C}$ are linearly dependent, as each side then reduces to $\vec{0}$.