In this brief note will will discuss various types of random variables. Throughout we are working with the general probability model $(\Omega, \mathcal{S}, \Pr)$. Recall that a function $X : \Omega \to \mathbf{R}$ is a random variable if and only if for every interval $\mathcal{I}$ in $\mathbf{R}$ we have

$$\{\omega \in \Omega : X(\omega) \in \mathcal{I}\} \in \mathcal{S}$$

1 Indicator Random Variables

A random variable X that only takes the values 0 and 1 is called an **indicator random variable**. Let $A = \{\omega \in \Omega : X(\omega) = 1\}$. The name indicator random variable is used because when $X = 1$ this "indicates" that ω is in A. Usually we write I_A for this random variable. Conversely, if an event B is given, we define its indicator, denoted I_B by the rule $I_B(\omega) = 1$ if $\omega \in B$ and $I_B(\omega) = 0$ if $\omega \in B^c$. Note that $1 - I_B = I_{B^c}$.

Indicator random variables are the simplest kind of random variable, and most other random variables are constructed in terms of them.

2 Simple Random Variables

A random variable X is called a **simple random variable** if there is a finite partition, $A_1, A_2, \ldots, A_N$ of Ω and constants $x_1, x_2, \ldots, x_N$ so that

$$X = x_1 I_1 + x_2 I_2 + \cdots + x_N I_N$$

where I_j is the indicator of A_j. Each indicator random variable is itself a simple random variable since we have

$$I_B = 1 I_B + 0 I_{B^c}.$$

In this definition of simple random variable there is no requirement that the values $x_1, x_2, \ldots, x_N$ are all different. This will be useful and important when we discuss expected value.

3 Lattice random variables

A lattice, L, in $\mathbf{R}$ is a set of equally spaced numbers, that is, there is some $x_0 \in \mathbf{R}$ and some $h \in \mathbf{R}$ so that

$$L = \{x \in \mathbf{R} : x = x_0 + kh, k \in \mathbf{Z}\}.$$

An example of a lattice is all the integer multiples of π.

A random variable is said to be a **lattice random variable** if its range is a subset of a lattice. A random variable may be simple, but not lattice. A random variable that has a binomial distribution is both lattice and simple.

4 Discrete random variables

A random variable X is said to be a **discrete random variable** if its range is a countable subset of $\mathbf{R}$. This does not mean the range of X is a discrete set. For example, an random variable whose range is the set of all positive rational number is a discrete random variable. However, all the types of random variables we have listed so far are discrete random variables. An important property of discrete random variables is that if X is a discrete random variable and R is its range then

$$\sum_{x \in R} \Pr(\{\omega \in \Omega : X(\omega) = x\}) = 1.$$

5 Continuous random variables

Recall that the distribution function of a random variable X, denoted by F_X is the function $F_X : \mathbf{R} \to [0, 1]$ defined by

$$F_X(t) = \Pr(\{\omega \in \Omega : X(\omega) \leq t\}).$$

The random variable X is said to be a **continuous random variable** if its distribution function is a continuous function. This is equivalent to requiring that for each real number t,

$$\Pr(\{\omega \in \Omega : X(\omega) = t\}) = 0.$$

Continuous random variables are the opposite of discrete random variables in that for a discrete random variable there are many values of t where

$$\Pr(\{\omega \in \Omega : X(\omega) = t\}) > 0.$$

6 Absolutely continuous random variables

A random variable X is said to be an **absolutely continuous random variable** if there is some function $f : \mathbf{R} \to [0, \infty)$ such that for every interval I,

$$\Pr(\{\omega \in \Omega : X(\omega) \in I\}) = \int_I f(x)\, dx.$$

The function f is called a **density function** for X.

For example, if X records the value of a real number chosen at random from the interval $[a, b]$ then a density function for X is the function f defined by $f(x) = 1/(b - a)$ if $x \in [a, b]$ and $f(x) = 0$ if $x < a$ or $x > b$.

Every absolutely continuous random variable is a continous random variable but not vice-versa. We shall not have any ocassion in this course to discuss continuous random variables that are not also absolutely continous random variables.

7 Independent random variables.

Two random variables X and Y defined on $(\Omega, \mathcal{S}, \Pr)$ are said to be **independent** if for every pair of intervals I and J

$$\Pr(\{\omega \in \Omega : X(\omega) \in I \text{ and } Y(\omega) \in J\}) = \Pr(\{\omega \in \Omega : X(\omega) \in I\}) \cdot \Pr(\{\omega \in \Omega : Y(\omega) \in J\})$$

Note that if X and Y are independent random variables, X is indicator of A with $0 < \Pr(A) < 1$ and Y is a continous random variable then XY is a random variable that is neither continuous nor discrete, since

$$\begin{aligned}
\Pr(\{\omega \in \Omega : X(\omega)Y(\omega) = 0\}) &= \Pr(\{\omega \in \Omega : X(\omega)Y(\omega) = 0 \text{ and } X(\omega) = 0\}) \\
&\quad + \Pr(\{\omega \in \Omega : X(\omega)Y(\omega) = 0 \text{ and } X(\omega) = 1\}) \\
&= \Pr(\{\omega \in \Omega : X(\omega) = 0\}) \\
&\quad + \Pr(\{\omega \in \Omega : Y(\omega) = 0 \text{ and } X(\omega) = 1\}) \\
&= \Pr(A) + \Pr(\{\omega \in \Omega : Y(\omega) = 0) \Pr(A^c) \\
&= \Pr(A) > 0
\end{aligned}$$

so that XY cannot be continuous, while if $t \neq 0$,

$$\begin{aligned}
\Pr(\{\omega \in \Omega : X(\omega)Y(\omega) = t\}) &= \Pr(\{\omega \in \Omega : X(\omega)Y(\omega) = t \text{ and } X(\omega) = 0\}) \\
&\quad + \Pr(\{\omega \in \Omega : X(\omega)Y(\omega) = t \text{ and } X(\omega) = 1\}) \\
&= \Pr(\{\omega \in \Omega : 0 \cdot Y(\omega) = t \text{ and } X(\omega) = 0\}) \\
&\quad + \Pr(\{\omega \in \Omega : Y(\omega) = 0 \text{ and } X(\omega) = 1\}) \\
&= \Pr(\emptyset) + \Pr(\{\omega \in \Omega : Y(\omega) = t) \Pr(A^c) \\
&= 0
\end{aligned}$$

so that XY cannot be discrete either because the sum of the probabilities that is equal to each of its range values is $\Pr(A) < 1$.