MthStat 465, Spring 2005, Lecture Number 22
Covariance and Variance of a Sum

In order to apply the Central Limit Theorem we need to be able to compute the variance and standard deviation of a sum of independent random variables.

We already know that if $\mathrm{E[R_k]} = \mu_\mathrm{k}$ then

$$\mathrm{E[R_1 + \cdots + R_N] = E[R_1] + \cdots + E[R_N]} \tag{1}$$

regardless of whether or not the random variables are independent. We also know that

$$\begin{aligned} \mathrm{Var[R_1 + \cdots + R_N]} &= \mathrm{E[((R_1 + \cdots + R_N) - (\mu_1 + \cdots + \mu_N))^2]} \\ &= \mathrm{E[((R_1 - \mu_1) + \cdots + (R_N - \mu_N))^2]} \\ &= \sum_{j,k} \mathrm{E[(R_j - \mu_j)(R_k - \mu_k)]} \end{aligned}$$

Now, $\mathrm{E[(R_j - \mu_j)(R_j - \mu_j)] = Var[R_j]}$. More generally, the quantity $\mathrm{E[(R_j - \mu_j)(R_k - \mu_k)]}$ is called the **covariance** of R_j and R_k, and is denoted by $\mathrm{Cov[R_j, R_k]}$.

Proposition 1. *If R and T are independent random variables which have expected values, then* $\mathrm{E[RT] = E[R]E[T]}$.

Proof. We give the proof in the case that the expected values can be computed with finite sums. The general case is based on this case.

Suppose that the range of R is $\{r_1, r_2, \ldots, r_N\}$ and the range of T is $\{t_1, t_2, \ldots, t_M\}$. Since R and T are independent we have $\Pr(\{R = x, T = y\}) = \Pr(\{R = x\}) \Pr(\{T = y\})$ for any x and y. Therefore

$$\begin{aligned} \mathrm{E[RT]} &= \sum_{j=1}^{N} \sum_{k=1}^{M} r_j t_k \Pr(\{R = r_j, T = t_k\}) \\ &= \sum_{j=1}^{N} \sum_{k=1}^{M} r_j t_k \Pr(\{R = r_j\}) \Pr(\{T = t_k\}) \\ &= \sum_{j=1}^{N} \left(r_j \Pr(\{R = r_j\}) \sum_{k=1}^{M} t_k \Pr(\{T = t_k\}) \right) \\ &= \sum_{j=1}^{N} \left(r_j \Pr(\{R = r_j\}) \mathrm{E[T]} \right) \\ &= \mathrm{E[T]} \sum_{\mathrm{j=1}}^{\mathrm{N}} \mathrm{r_j} \Pr(\{\mathrm{R = r_j}\}) \\ &= \mathrm{E[T]E[R]} \end{aligned}$$

so $\mathrm{E[RT] = E[R]E[T]}$. □

It follows from this proposition that if R_1 and R_2 are independent and have expected values then their covariance is 0:

$$\begin{aligned}
\mathrm{Cov(R_1, R_2)} &= \mathrm{E[(R_1 - E[R_1])(R_2 - E[R_2])]} \\
&= \mathrm{E[R_1R_2] - E[R_1]E[R_2] - E[R_2]E[R_1] + E[R_2]E[R_1]} \\
&= \mathrm{E[R_1]E[R_2] - E[R_1]E[R_2] - E[R_2]E[R_1] + E[R_2]E[R_1]} \\
&= 0
\end{aligned}$$

We have proven the following:

Proposition 1. *If* $R_1, R_2, \ldots, R_N$ *are independent with* $\mathrm{Var[R_j]} = \sigma_{\mathrm{j}}^2$ *then*

$$\mathrm{Var[R_1 + \cdots + R_N]} = \sigma_{\mathrm{1}}^2 + \cdots + \sigma_{\mathrm{N}}^2.$$

In particular, if the random variables all have the same distribution then there is some $\sigma^2 = \sigma_j^2$ *for all* j*, and*

$$\mathrm{Var[R_1 + \cdots + R_N]} = \mathrm{N}\sigma_{\mathrm{1}}^2.$$

This generalizes the result for binomial random variables that says that the variance for the number of heads in N tosses of a coin is $Np(1-p)$.