MthStat 465, Spring 2005, Lecture Number 13
Density Functions.

Given a random variable R and its distribution function F_R, we say that f_R is a probability density function for R if for every real number t,

$$F_R(t) = \int_{-\infty}^{t} f_R(u)\, du.$$

Any $f : (-\infty, \infty) \to [0, \infty)$ is a density function so long as

$$\int_{-\infty}^{\infty} f(u)\, du = 1.$$

For example, if

$$f(u) = \begin{cases} 0 & \text{if} \quad u < -1 \\ 1/3 & \text{if} \quad -1 \leq u \leq 2 \\ 0 & \text{if} \quad u > 2 \end{cases}$$

then f is a density function (graph it and use area of a rectangle!), and the corresponding distribution function is

$$F(u) = \begin{cases} 0 & \text{if} \quad u \leq -1 \\ 1/3(t+1) & \text{if} \quad -1 \leq u \leq 2 \\ 1 & \text{if} \quad u \geq 2 \end{cases}$$

The random variable is the value of a real number chosen at random in $[-1, 2]$.

Perhaps the most famous density is the so-called **bell-shaped curve**, officially known as the **standard normal density**. It is given by

$$f(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right).$$

It is clear that this function is never negative, but not so clear why its integral is 1. Here is an argument to show this.

It is sufficient to show that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right)\, du = \sqrt{(2\pi)}.$$

Denote this integral by I. We have

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) \exp\left(-\frac{x^2}{2}\right) dydx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) dydx \\ &= \int_{0}^{2\pi} \int_{0}^{\infty} \exp\left(-\frac{r^2}{2}\right) rdrd\theta \\ &= 2\pi. \end{aligned}$$

There are many other densities that have the same general shape as the standard normal density. For example, $f(u) = \pi^{-1}(1+u^2)^{-1}$. It is important not to conclude that that your data has a standard normal density just because its histogram is bell-shaped.