**MthStat 465, Spring 2005, Lecture Number 12**
**A coin tossing model.**

We give a complete model of tossing a coin once under the assumption that the tosses are independent. To keep things concrete, we will toss the coin 20 times. The generalization to $N$ times will be clear.

First we make the assumption that the only possibilities for each toss are Head and Tail, which we denote by H and T respectively. We will take the sample space to be the set of all sequences of length 20 drawn from the set $\{H, T\}$. The $k^{\text{th}}$ term in the sequence denotes the result of the $k^{\text{th}}$ toss. For example, $(T, T, T, H, H, T, H, T, H, T, T, H, H, H, H, T, H, H, T, T)$ is one possible outcome. There are $2^{20} = 1,048,576$ outcomes.

Since the sample space is finite, there is no harm in letting the sigma algebra be the set of all subsets of the sample space. The sigma algebra contains

$$2^{1,048,576} \approx 10^{315,652.8287}$$

elements, each one a subset of the sample space!

To assign probablities, it would be sufficient to find a sequence $(p_0, p_1, \ldots, p_{2^{20}-1}$ of non-negative real numbers with

$$\sum_{k=0}^{2^{20}-1} p_k = 1,$$

order the outcomes from 0 to $2^{20} - 1$ by letting each outcome represent a binary integer, and the define

$$\Pr(E) = \sum_{k \in E} p_k.$$

However, we want certain events to be independent, and independence depends solely on how the probabilities are defined, so we don't have a free hand.

We will do the following. Pick $p \in [0, 1]$, and for each single element event $s$ we will define

$$\Pr(\{s\}) = p^{\text{number of H's in } s}(1-p)^{\text{number of T's in } s}$$

and then define the probability of any other event $F$ to be the sum of the probabilities of the single element events whose union is the event $F$. (Note that if $p = 1/2$ then we are saying that all outcomes are equally likely, and we would say that the coin was fair.)

For example, if $F = \{10 \text{ Heads}, 10 \text{ Tails}\}$, each element in the event has probability $p^{10}(1-p)^{10}$ and there are ${}_{20}C_{10}$ such elements, so

$$\Pr(F) =_{20} C_{10} p^{10}(1-p)^{10}$$

More generally, if $F = \{k \text{ Heads}, 20 - k \text{ Tails}\}$,

$$\Pr(F) =_{20} C_k p^k (1-p)^{20-k}$$

By dividing the sample space up into subsets containing outcomes with a given number of heads, we see that

$$\begin{aligned}
\Pr(S) &= \sum_{k=0}^{20} \Pr(\{k \text{ Heads}, 20-k \text{ Tails}\}) \\
&= \sum_{k=1}^{20} {}_{20}C_k p^k (1-p)^{20-k} \\
&= (p + (1-p))^{20} \\
&= 1
\end{aligned}$$

so we have a proper assignment of probabilities.

We can define random variables in our model. Since the sigma algebra is the set of all subsets of $S$, every function defined on $S$ is a random variable. For example, let $R$ be the function that counts the number of H's in s. Then

$$\Pr(R = k) =_{20} C_k p^k (1-p)^{20-k}$$

if $k$ is a non-negative integer, and 0 otherwise.

Finally, we can check that $\Pr(\{\text{First toss is Heads}\}) = p$ by considering how many heads are in the remaining 19 tosses.