**MthStat 465, Spring 2005, Homework Number 5**

1. Give the defintions of Type I and Type II error in the hypothesis testing.

   A **Type I** error is made when one says that the data does not support the null hypothesis when the null hypothesis is true.

   A **Type II** error is made when one says that the data supports the null hypothesis when the null hypothesis is false.

2. Devise at test to distinguish the following two possibilities:
   - A coin has probability 0.45 of coming up heads on any one toss, and all tosses of the coin are independent;
   - A coin has probability 0.60 of coming up heads on any one toss, and all tosses of the coin are independent.

   Your test should be right at least 90% of the time in either case. Check your results against the formulae derived in Lecture 19. Do not simply "plug in" to the formulae we derived there. Verify that your tests meet the objectives, and adjust them if need be.

   Let $H_N$ be the number of heads obtained in $N$ independent tosses of this coin. If $H_N \leq C$ we will say that the coin comes up heads with probability 0.45 and if $H_N > C$ we will say that the coin comes up heads with probability 0.60. This seems reasonable as a greater number of heads is associated with a higher probability of obtaining a head on any one toss. We need to determine both $N$ and $C$. No matter which case we have the probability that $H_N = k$ is given by a binomial distribution. Therefore we want:

$$\begin{aligned} \sum_{k=0}^{C} \binom{N}{k} (0.45)^k (0.55)^{N-k} &\geq 0.90 \\ \sum_{k=C+1}^{N} \binom{N}{k} (0.60)^k (0.40)^{N-k} &\geq 0.90 \end{aligned}$$

   Each of these sums can be approximated by the DeMoivre-Laplace theorem:

$$\begin{aligned} \sum_{k=0}^{C} \binom{N}{k} (0.45)^k (0.55)^{N-k} &\approx \int_{-\infty}^{(C-0.45N)/\sqrt{0.45\times 0.55N}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du; \\ \sum_{k=C+1}^{N} \binom{N}{k} (0.60)^k (0.40)^{N-k} &\approx \int_{(C-0.60N)/\sqrt{0.60\times 0.40N}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du; \end{aligned}$$

   We know that

$$\int_{-1.2816}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = \int_{-\infty}^{1.2816} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \approx 0.9000084999$$

   so to get approximate values of $C$ and $N$ we can solve the system of equations

$$\begin{aligned} \frac{C-0.45N}{\sqrt{0.45\times 0.55N}} &= 1.2816 \\ \frac{C-0.60N}{\sqrt{0.60\times 0.40N}} &= -1.2816 \end{aligned}$$

We get $C \approx 37.4$ and $N \approx 71.2$. We must choose $C$ and $N$ to be integers, so we will try $N = 72$ and $C = 38$. Putting these values back into the sums:

$$\sum_{k=0}^{38} \binom{72}{k} (0.45)^k (0.55)^{72-k} \approx .9254290848$$

$$\sum_{k=39}^{72} \binom{72}{k} (0.60)^k (0.40)^{72-k} \approx .8705313459$$

which isn't quite good enough in the second sum. A little experimentation shows that the smallest $N$ we can use is $N = 77$ and we need $C = 40$:

$$\sum_{k=0}^{40} \binom{77}{k} (0.45)^k (0.55)^{77-k} \approx .9095828001$$

$$\sum_{k=41}^{77} \binom{77}{k} (0.60)^k (0.40)^{77-k} \approx .9068049208$$

Notice that we don't bother with the continuity correction here because we are only trying to approximate $C$ and $N$.

3. When testing hypotheses in coin tossing where the unknown quantity is $p$, the probability of heads, what is meant by a "one sided alternative"? What is meant by a "two-sided alternative"? Illustrate by constructing three tests where there are 100 tosses, the null hypothesis is that $p = 0.4$, and the probability of Type I error is no more than 0.01.
4. Toss a penny 100 times. Test the hypothesis that the probability of heads is 0.45 against the alternative that $p = 0.5$ when the probability of Type I error is not more than
   - 0.10
   - 0.05
   - 0.01.

   What is the $p$-value for your data?
5. Using the same data as in the previous problem, test the hypothesis that the probability of heads is 0.5 against the alternative the $p = 0.45$ with the same probabilities of Type I error. What is the $p$-value for your data?
6. Comment on the results of the last two exercises.
7. Suppose we are testing for the probability of heads. How will the $p$-value of our test change if we switch from a one-sided alternative to a two sided alternative. Explain, and use examples.