# USING EIGENVECTORS TO SOLVE A SYSTEM OF ODE'S
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### 1. Introduction

Step by step instructions for solving a constant coefficient linear system of differential equations if the set of eigenvectors of the coefficient matrix forms a basis.

### 2. The problem

Assume $\vec{x} : \mathbf{R} \to \mathbf{R}^n$ and $\vec{g} : \mathbf{R} \to \mathbf{R}^n$ are vectors of functions, and $M$ is an $n \times n$ matrix of real numbers. Assume that $\vec{x}(0)$ and $\vec{g}(t)$ are given, and

$$\frac{d}{dt}\vec{x}(t) + M\vec{x}(t) = \vec{g}(t). \tag{1}$$

In addition (this is a big assumption, and not always satisfied), assume that there is a basis of $\mathbf{R}^n$ consisting of eigenvectors of $M$. The following steps will lead to a solution of (1).

We will illustrate the general method with this example as we go along:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} t \\ \sin(t) \end{bmatrix} \tag{2}$$

and $x_1(0) = -2$, $x_2(0) = 2$.

### 3. Step 1: Find the eigenvalue/eigenvector pairs for $M$.

Label the eigenvalue/eigenvector pairs $(s_1, \vec{v}_1), (s_2, \vec{v}_2), \ldots, (s_n, \vec{v}_n)$, so we have

$$M\vec{v_k} = s_k\vec{v}_k. \tag{3}$$

In the example, the characteristic polynomial is $p(t) = (t-2)(t-4) - 3 = (t-1)(t-5)$, giving eigenvalues of 1 and 5. An eigenvector for 1 is $[-3, 1]^t$ and for 5 we can use $[1, 1]^t$:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -3 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

### 4. Step 2: Convert all the vectors to linear combinations of eigenvectors

Since we have a basis of eigenvectors, there are functions $f_1, f_2, \ldots, f_n$ so that

$$\vec{x}(t) = f_1(t)\vec{v}_1 + f_2(t)\vec{v}_2 + \cdots f_n(t)\vec{v}_n, \tag{4}$$

functions $h_1, h_2, \ldots, h_n$ so that

$$\vec{g}(t) = h_1(t)\vec{v}_1 + h_2(t)\vec{v}_2 + \cdots h_n(t)\vec{v}_n, \tag{5}$$

and scalars functions $y_1, y_2, \dots, y_n$ so that

$$\vec{x}(0) = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots y_n\vec{v}_n. \tag{6}$$

The functions $h_1(t), h_2(t), \dots, h_n(t)$ and the scalars $y_1, y_2, \dots, y_n$ have to be solved for from the given components of $\vec{g}$ and $\vec{x}(0)$ respectively, and (5) and (6).

In our example we have

$$\begin{aligned}
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= f_1(t)\begin{bmatrix} -3 \\ 1 \end{bmatrix} + f_2(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} t \\ \sin(t) \end{bmatrix} &= h_1(t)\begin{bmatrix} -3 \\ 1 \end{bmatrix} + h_2(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} &= y_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} + y_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

Solving the set of equations for $h_1(t)$ and $h_2(t)$ gives $h_1(t) = (\sin(t) - t)/4$ and $h_2(t) = (3\sin(t) + t)/4$. Solving the equations for $y_1$ and $y_2$ gives $y_1 = 1$ and $y_2 = 1$, so we have

$$\begin{aligned}
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= f_1(t)\begin{bmatrix} -3 \\ 1 \end{bmatrix} + f_2(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} t \\ \sin(t) \end{bmatrix} &= \frac{1}{4}(\sin(t) - t)\begin{bmatrix} -3 \\ 1 \end{bmatrix} + \frac{1}{4}(3\sin(t) + t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} &= 1\begin{bmatrix} -3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

## 5. Step 3: Substitute the eigenvector representations into the differential equation and simplify

We now substitute (5) and (6) into the differential equation (1) and use the eigenvalue/eigenvector property to simplify:

$$\begin{aligned}
\frac{d}{dt}\vec{x}(t) &= f_1'(t)\vec{v}_1 + f_2'(t)\vec{v}_2 + \cdots + f_n'(t)\vec{v}_n. \\
M\vec{x}(t) &= f_1(t)M\vec{v}_1 + f_2(t)M\vec{v}_2 + \cdots + f_n(t)M\vec{v}_n \\
&= s_1f_1(t)\vec{v}_1 + s_1f_1(t)\vec{v}_1 + \cdots + s_1f_1(t)\vec{v}_1 \\
\vec{g}(t) &= h_1(t)\vec{v}_1 + h2(t)\vec{v}_2 + \cdots + h_n(t)\vec{v}_n
\end{aligned}$$

so (1) becomes

$$\begin{aligned}
& (f_1'(t) + s_1f_1(t))\vec{v}_1 + (f_2'(t) + s_2f_2(t))\vec{v}_2 + \cdots + (f_n'(t) + s_nf_n(t))\vec{v}_n \\
= \; & h_1(t)\vec{v}_1 + h_2(t)\vec{v}_2 + \cdots + h_n(t)\vec{v}_n
\end{aligned}$$

or, collecting like terms:

$$\begin{aligned}
(f_1'(t) + s_1f_1(t) - h_1(t))\vec{v}_1 + (f_2'(t) + s_2f_2(t) - h_2(t))\vec{v}_2 & \\
+ \cdots + (f_n'(t) + s_nf_n(t) - h_n(t))\vec{v}_n & \\
&= \vec{0}.
\end{aligned}$$

Since the eigenvectors are linearly independent, we have

$$\begin{aligned} f_1'(t) + s_1 f_1(t) - h_1(t) &= 0 \\ f_2'(t) + s_2 f_2(t) - h_2(t) &= 0 \\ &\vdots \\ f_n'(t) + s_n f_n(t) - h_n(t) &= 0 \end{aligned}$$

Furthermore, plugging $t = 0$ into (5) and comparing with (6) shows us that

$$\begin{aligned} f_1(0) + s_1 f_1(t) - h_1(t) &= y_1 \\ f_2(0) + s_2 f_2(t) - h_2(t) &= y_1 \\ &\vdots \\ f_n(0) + s_n f_n(t) - h_n(t) &= y_n \end{aligned}$$

In terms of our example, the differential equation (1) becomes

$$\begin{aligned} & f_1'(t) \begin{bmatrix} -3 \\ 1 \end{bmatrix} + f_2'(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ & + f_1(t) \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + f_2(t) \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ = \; & \frac{1}{4}(\sin(t) - t) \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \frac{1}{4}(3\sin(t) + t) \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}. \end{aligned}$$

Using that the vectors are eigenvectors for the matrix,

$$\begin{aligned} & f_1'(t) \begin{bmatrix} -3 \\ 1 \end{bmatrix} + f_2'(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ + f_1(t) \begin{bmatrix} -3 \\ 1 \end{bmatrix} + 5 f_2(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{4}(\sin(t) - t) \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \frac{1}{4}(3\sin(t) + t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

and collecting like terms,

$$(f_1'(t) + f_1(t) - \frac{1}{4}(\sin(t) - t)) \begin{bmatrix} -3 \\ 1 \end{bmatrix} + f_2'(t) + f_2(t) - \frac{1}{4}(3\sin(t) + t)) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Finally, $f_1(0) = 1$ and $f_2(0) = 1$.