

Increasing, decreasing, convex and concave functions

We investigate the relation between derivatives and the shape of graphs. The basic result is called **Rolle's Theorem**.

1 Rolle's Theorem

Throughout this section we assume $a < b$.

Theorem 1 (Rolle's Theorem) *Suppose that f is a continuous function with domain $[a, b]$, that $f(a) = f(b)$ and f is differentiable at every $x \in (a, b)$. Then there is at least one $c \in (a, b)$ where $f'(c) = 0$.*

Reason: There are two possibilities for f . The first is that f is constant. Then $f'(c) = 0$ for every $c \in (a, b)$. The second possibility is that f is not constant. Then it has an absolute maximum $(M, f(M))$, an absolute minimum $(m, f(m))$ and $f(m) < f(M)$. This means either $M \in (a, b)$ or $m \in (a, b)$. If the former, then $f'(M) = 0$ since f is differentiable at M . If the latter, then $f'(m) = 0$ since f is differentiable at m . **QED**

Note: The point of Rolle's theorem is that such a number c exists. We have no interest in the value of c .

2 The First Mean Value Theorem

Theorem 2 (The First Mean Value Theorem) *If g is a continuous function with domain $[a, b]$ and g is differentiable at each $x \in (a, b)$. Then there is at least one $c \in (a, b)$ so that $g(b) - g(a) = g'(c)(b - a)$.*

Reason: Define $f(x) = (g(x) - g(a))(b - a) - (x - a)(g(b) - g(a))$. f is the sum of products of functions continuous on $[a, b]$ and differentiable on (a, b) , with

$$f'(x) = (g'(x) - 0)(b - a) - (1 - 0)(g(b) - g(a)) = g'(x)(b - a) - (g(b) - g(a)).$$

Furthermore, $f(a) = f(b) = 0$, so we may conclude from Rolle's Theorem that there is some $c \in (a, b)$ where

$$0 = f'(c) = g'(c)(b - a) - (g(b) - g(a)),$$

which is the same as $g(b) - g(a) = g'(c)(b - a)$. **QED**

Again, the point of this theorem is that such a c exists, not to calculate it. For example:

Corollary 1 *Suppose that the domain of f is I , is an interval with endpoints $a < b$, that f is continuous on I , and f is differentiable on (a, b) . Then*

- If $f'(x) = 0$ for each $x \in (a, b)$ then f is constant on I .
- If $f'(x) > 0$ for each $x \in (a, b)$ then f is strictly increasing on I .
- If $f'(x) \geq 0$ for each $x \in (a, b)$ then f is non-decreasing on I .
- If $f'(x) < 0$ for each $x \in (a, b)$ then f is strictly decreasing on I .
- If $f'(x) \leq 0$ for each $x \in (a, b)$ then f is non-increasing on I .

Reason: Each case is similar. Throughout what follows, u and v are elements of I and $u < v$. Remember that differentiability at a point implies continuity at that point. The First Mean Value Theorem tells us that for some $u < c < v$ that

$$f(v) - f(u) = f'(c)(v - u).$$

- If $f'(x) = 0$ for all $a < b < x$ then

$$f(v) - f(u) = f'(c)(v - u) = 0$$

so $f(v) = f(u)$ for any u and v in the domain of f .

- If $f'(x) > 0$ for all $a < b < x$ then

$$f(v) - f(u) = f'(c)(v - u) > 0$$

so $f(v) > f(u)$ for any u and v in the domain of f .

- If $f'(x) \geq 0$ for all $a < b < x$ then

$$f(v) - f(u) = f'(c)(v - u) \geq 0$$

so $f(v) \geq f(u)$ for any u and v in the domain of f .

- If $f'(x) < 0$ for all $a < b < x$ then

$$f(v) - f(u) = f'(c)(v - u) < 0$$

so $f(v) < f(u)$ for any u and v in the domain of f .

- If $f'(x) \leq 0$ for all $a < b < x$ then

$$f(v) - f(u) \leq f'(c)(v - u) = 0$$

so $f(v) \leq f(u)$ for any u and v in the domain of f .

Note that the first assertion implies that if two functions agree at one point of an interval and their derivatives agree at all points of the interval, then the functions agree on the interval. This extends the idea that parallel lines that cross are the same line, since parallel lines have the same slope, and the derivative gives the slope at a point.

2.1 Exponential Growth and Decay

We learned in algebra that in certain situations a function of the form $f(t) = A \exp(rt)$ describes the phenomenon in question. For example, suppose that at time 0 a Petri dish has a colony consisting of 10 micrograms of *Listeria* bacteria, and 5 hours later there are 13 micrograms of this bacteria. How much bacteria is present at time t hours.

The underlying principle in these problems was that the instantaneous rate of change of the quantity in question was assumed to be proportional to the amount of the quantity present. We can now express this concisely using derivatives:

$$\frac{d}{dt}Q(t) = rQ(t)$$

where $Q(t)$ is the amount of the quantity in question, and r is the proportionality constant. If we suppose that $Q(0) \neq 0$ is given then we can use the Mean Value Theorem to show that the only solution to our problem is $Q(t) = Q(0) \exp(rt)$.

It is clear from the Chain Rule that $Q(t)$ is one solution, since

$$\frac{d}{dt}Q(t) = \frac{d}{dt}Q(0) \exp(rt) = Q(0) \frac{d}{dt} \exp(rt) = Q(0) \exp(rt) \frac{d}{dt}(rt) = Q(0) \exp(rt) \cdot r = rQ(t)$$

and $Q(0) = Q(0) \exp(0)$. Suppose now that f is another solution, that is $f(0) = Q(0)$ and $f'(t) = rf(t)$. We will show that $f(t) - Q(t) = 0$ for all t . To do this, put $G(t) = f(t) - Q(t)$. We know that $G(0) = f(0) - Q(0) = 0$. We will now show that $G(t) = 0$ for any t by showing $G'(t) = 0$. Observe that

$$G'(t) = f'(t) - Q'(t) = rf(t) - rQ(t) = rG(t).$$

We have the following equivalent statements

$$\begin{aligned}G'(t) &= rG(t) \\G'(t) - rG(t) &= 0 \\ \exp(rt)(G'(t) - rG(t)) &= 0 \\ \exp(rt)G'(t) - r\exp(rt)G(t) &= 0 \\ (\exp(rt)G(t))' &= 0 \text{ Product Rule!}\end{aligned}$$

This last statement is equivalent to $\exp(rt)G(t)$ being a constant function. Since the value of $\exp(rt)G(t)$ is 0 when $t = 0$, $\exp(rt)G(t) = 0$ for any t . Since $\exp(rt) > 0$, it must be that $G(t) = 0$ for any t .

2.2 Another look at Hooke's Law

We examined the situation where we wanted to find a function f with the following properties:

$$\begin{aligned}f''(t) &= -r^2f(t) \\ f(0) &= A \\ f'(0) &= B\end{aligned}$$

where $r \neq 0$.

We observed that $f(t) = A \cos(rt) + B \sin(rt)$ fits the bill. We will now show that this is the only such function. Suppose that we also have

$$\begin{aligned}g''(t) &= -r^2g(t) \\ g(0) &= A \\ g'(0) &= B.\end{aligned}$$

We will show that $f(t) - g(t) = 0$ for any t . Let $h(t) = f(t) - g(t)$. It is easy to check that

$$\begin{aligned}h''(t) &= -r^2h(t) \\ h(0) &= 0 \\ h'(0) &= 0.\end{aligned}$$

Now put $E(t) = (rh(t))^2 + (h'(t))^2$. We have $E(0) = 0^2 + 0^2 = 0$ and

$$\frac{d}{dt}E(t) = 2(rh(t))rh'(t) + 2h'(t)h''(t) = 2h'(t)(r^2h(t) + h''(t)) = 0.$$

Therefore $E(t) = 0$ for every t . We then have $0 \leq (rh(t))^2 \leq E(t) = 0$, so $h(t) = 0$ for all t . Hence the **only** solution to our problem was $f(t) = A \cos(rt) + B \sin(rt)$.