Limits of Quotients

1 Why division by zero cannot be defined

When we are asked to divide a by b we are asked to find a number c so that $a = bc$, and we ask that there should be exactly one such number c. We also recognize that $0 \times c = 0$ whatever c might be. So here is why we cannot make sense of division by 0.

Under no circumstances should we ever write $a/0$ or $a \div 0$, no matter what we may see in textbooks.

1.1 The case of $a \neq 0$

If we were to be able to divide a non-zero number a by 0 we would have to find a number c so that $a = c \times 0$. Since $c \times 0 = 0$ for any choice of c, and $a \neq 0$, there can be no such number c.

1.2 The case of $a = 0$

In the previous case, we had no candidate for the result of dividing a by 0. This time the problem is that we have more than one choice. To be able to divide 0 by 0 we have to find exactly one number c so that $0 = c \times 0$. Now any number c will do.

2 Limits of Quotients

We wish to consider the possibility of $f(x)/g(x)$ having a limit as x approaches a. In the discussion that follows it does not matter if a is a real number or if a is the symbol $-\infty$ or $+\infty$. These same arguments apply to one-sided limits. There are several cases to consider:

Case I: This is the simplest case:

$$\begin{aligned} \lim_{x \to a} f(x) &= F \\ \lim_{x \to a} g(x) &= G \neq 0 \end{aligned}$$

We have shown that in this case

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}.$$

Case II:

$$\begin{aligned} \lim_{x \to a} f(x) &= 0 \\ |g(x)| &\geq G > 0. \end{aligned}$$

Then

$$0 \leq \left| \frac{f(x)}{g(x)} \right| \leq \frac{|f(x)|}{G}$$

so it follows from the Pinching Principle that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

For example, consider the quotient $\sin(x)/(2 + \cos(x))$ as x approaches 0.

Case III:

$$\begin{aligned} |f(x)| &\leq F \\ \lim_{x \to a} g(x) &= \infty \end{aligned}$$

or

$$\begin{aligned} |f(x)| &\leq F \\ \lim_{x \to a} g(x) &= -\infty \end{aligned}$$

Then

$$\lim_{x \to a} |g(x)| = \infty.$$

Since

$$0 \leq \left| \frac{f(x)}{g(x)} \right| \leq \frac{F}{|g(x)|}$$

it follows from the Pinching Principle that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

For example, consider the quotient $\sin(x)/x$ as x approaches infinity.

Case IV:

$$\begin{aligned} \lim_{x \to a} f(x) &= F \neq 0 \\ \lim_{x \to a} g(x) &= 0 \end{aligned}$$

In this case there cannot be a limit. The argument is similar to showing that we cannot define division of a non-zero number by 0. Suppose there were some number L so that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Then

$$F = \lim_{x \to a} f(x) = \lim_{x \to a} \left(g(x) \frac{f(x)}{g(x)} \right) = \lim_{x \to a} g(x) \lim_{x \to a} \frac{f(x)}{g(x)} = 0 \times L = 0$$

which is not possible.

Case V:

$$\begin{aligned} |f(x)| &\geq F > 0 \\ \lim_{x \to a} g(x) &= 0 \end{aligned}$$

Again, there can be no limit. For suppose that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Then

$$\lim_{x \to a} g(x) \frac{|f(x)|}{g(x)} = \lim_{x \to a} \frac{g(x)}{\times} \lim_{x \to a} g(x) \frac{|f(x)|}{g(x)} = 0 \times L = 0$$

which contradicts

$$0 < F \leq |f(x)| = g(x) \frac{|f(x)|}{g(x)}.$$

For example, consider the quotient $(2 + \sin(x))/x$ as x approaches 0.

Case VI: This is frequently called the **indeterminate case**:

$$\begin{aligned} \lim_{x \to a} f(x) &= 0 \\ \lim_{x \to a} g(x) &= 0 \end{aligned}$$

In this case we can draw no conclusion without specific information about f and g. Just consider these five examples, all with $a = 0$:

- $|x|/x$. The left and righthand limits exist, but are different.
- $\sin(x)/x$. The limit is 1.
- $\sin^2(x)/x$. The limit is 0.
- $\sin(x)/x^2$. The limit is undefined.
- $\sin(x)^2/x^4$. The limit is undefined, and specifically is said to be infinite.