Maxima and Minima

We investigate the relation between derivatives and extreme values of functions.

1 Minima and Maxima

The point $(M, f(M))$ is called an **absolute maximum** of f if $f(x) - f(M) \leq 0$ for every x in the domain of f. The point $(m, f(m))$ is called an **absolute minimum** of f if $f(x) - f(m) \geq 0$ for every x in the domain of f.

More than one absolute maximum or minimum may exist. For example, if $f(x) = |x|$ for $x \in [-1, 1]$ then $f(x) \leq 1$ and there are absolute maxima at $(1, 1)$ and at $(-1, 1)$, but only one absolute minimum, at $(0, 0)$.

Note also that if $(M, f(M)$ is an absolute maximum for f then $(M, -f(M))$ is an absolute minimum for $-f$, and if $(m, f(m)$ is an absolute minimum for f then $(m, -f(m))$ is an absolute maximum for $-f$,

Recall that if f is a continuous function with domain $[a, b]$ then there is some $M \in [a, b]$ such that $f(x) \leq f(M)$ for all $x \in [a, b]$ and there is some $m \in [a, b]$ such that $f(x) \geq f(m)$. In fact, what we know is that the range of f is $[f(m), f(M)]$.

We want to see what additional information the differentiability of f gives us.

2 A convention to remember

If the domain of f is $[a, b]$ and

$$\frac{f(x) - f(a)}{x - a}$$

has a limit as x approaches a from above, then we say that f is differentiable at a and we write $f'(a)$ for the derivative. Similarly, if

$$\frac{f(x) - f(b)}{x - b}$$

has a limit as x approaches b from below, then we say that f is differentiable at b and we write $f'(b)$ for the derivative.

3 Derivatives, Maxima and Minima

The following are the important observations from which all the rest of our results will follow.

Theorem 1 *Suppose that the domain of f is an interval I with end points $a < b$. Suppose that $c \in I$ and $f'(c) = d \neq 0$. If $a < c < b$ then $(c, f(c))$ is neither an absolute maximum nor an absolute minimum.*

Reason: The argument hinges on the following observation. If

$$\lim_{x \to u} g(x) = G > 0$$

then there is some $d > 0$ so that if x is in the domain of g and $0 < |x - u| < d$ then $g(x) > 0$. To see why, apply the definition of limit with the tolerance $t = G/2$.

Suppose first that $f'(c) > 0$. Then there is some $u > c$ so that

$$\frac{f(u) - f(c)}{u - c} > 0.$$

Since $u - c > 0$ we know that $f(u) - f(c) > 0$, so $(c, f(c))$ is not an absolute maximum. There is also some $v < c$ so that

$$\frac{f(v) - f(c)}{v - c} > 0.$$

Since $v - c < 0$ we know that $f(v) - f(c) < 0$. This means $(c, f(c))$ is not an absolute minimum.

The other possibility is that $f'(c) < 0$. Just apply the preceding argument to $-f$. **QED**

The preceding takes care of points that are not endpoints of the domain of a function. To investigate the behavior at endpoints of the domain we have to consider several cases. None are complicated, but we have to cover all the bases.

Theorem 2 *Suppose that the domain of f is an interval I with end points $a < b$.*

Case A: *Suppose that $a \in I$. If $f'(a) > 0$ then $(a, f(a))$ is not an absolute maximum. If $f'(c) < 0$ then $(c, f(c))$ is not an absolute minimum.*

Case B: *Suppose $b \in I$. If $f'(b) < 0$ then $(b, f(b))$ is not an absolute maximum. If $f'(b) > 0$ then $(b, f(b))$ is not an absolute minimum.*

Reason: The arguments are similar to the preceding theorem.

In **Case A** suppose that $f'(a) > 0$. This means that there is some $u > a$ so that

$$\frac{f(u) - f(a)}{u - a} > 0.$$

Since $u - a > 0$ we know that $f(u) - f(a) > 0$. This means that $(a, f(a))$ is not an absolute maximum. If $f'(a) < 0$ apply the preceding argument to $-f$.

In **Case B** suppose that $f'(b) > 0$. This means there is some $u < b$ so that

$$\frac{f(u) - f(b)}{u - b} > 0.$$

Since $u - b < 0$ we have $f(u) - f(b) < 0$ so $(b, f(b))$ cannot be an absolute minimum. If $f'(b) < 0$ apply the preceding argument to $-f$. **QED**

Corollary 1 *Suppose that f is a continuous function with domain $[a, b]$. If $(M, f(M))$ is an absolute maximum, then one of the following is true.*

- *$M = a$;*
- *$M = b$;*
- *f is not differentiable at M;*
- *f is differentiable at M and $f'(M) = 0$.*

If $(m, f(m))$ is an absolute minimum, then one of the following is true.

- *$m = a$;*
- *$m = b$;*
- *f is not differentiable at m;*
- *f is differentiable at m and $f'(m) = 0$.*

Reason: The only possibility not addressed is that f is differentiable at M or m with a non-zero derivative. In that case we have neither an absolute maximum nor an absolute minimum. **QED**

3.1 A typical application

Suppose that $f(x) = 2x^3 - 3x^2 + 2$ with domain $[-2, 2]$. Locate the absolute maxima and absolute minima.

Solution: Since f is continuous and its domain is a closed interval, there is at least one absolute maximum and one absolute minimum. f is differentiable on its entire domain, with $f'(x) = 6x^2 - 6x = 6x(x - 1)$. So the possible location of the absolute maxima and minima are

$$\{(-2, f(-2)), (0, f(0)), (1, f(1)), (2, f(2))\} = \{(-2, -26), (0, 2), (1, 1), (2, 6)\}$$

from which it is clear that the absolute maximum is at $(2, 6)$ and the absolute minimum is at $(-2, -26)$.

3.2 An important example

Suppose that $f(x) = |x|$ with domain $[-2, 3]$. We know that the absolute maximum is $(3, 3)$ and the absolute minimum is $(0, 0)$. Notice that the minimum occurs at a place where the derivative is undefined.

3.3 A geometry problem

Consider the graph of $y = x^2$, for $x \in [-2, 2]$. What point on the graph is closest to $(0, 1)$ and which point is furthest away?
Solution: The distance from $(0, 1)$ to (x, x^2) is $\sqrt{(x-0)^2 + (x^2-1)^2}$. So we want to find the absolute maximum and minimum of the function $f(x) = \sqrt{(x-0)^2 + (x^2-1)^2}$ with domain $[-2, 2]$. f is continuous, so there is an absolute maximum and an absolute minimum, while

$$f'(x) = \frac{1}{2}(x^2 + (x^2-1)^2)^{-1/2}\left(2x + 2(x^2-1)(2x)\right) = \frac{x(2x^2-1)}{\sqrt{x^2+(x^2-1)^2}}$$

for every $x \in [-2, 2]$. Therefore the only possible locations for the absolute maxima are

$$\begin{aligned} & \{(-2, f(-2)), (-1/\sqrt{2}, f(-1/\sqrt{2}), (0, f(0)), (1/\sqrt{2}, f(1/\sqrt{2}), (2, f(2))\} \\ = \; & \{(-2, \sqrt{13}), (-1/\sqrt{2}, \sqrt{3}/2), (0, 1), (1/\sqrt{2}, \sqrt{3}/2), (2, \sqrt{13})\} \end{aligned}$$

so we see the closest points are $(-1/\sqrt{2}, 1/2)$ and $(1/\sqrt{2}, 1/2)$, and the most distant points are $(-2, 4)$ and $(2, 4)$.