More special limits

1 Rational Powers

Theorem 1 *Suppose that q is a positive integer and p is any integer and $r = p/q$. Then*

$$\begin{aligned}
\lim_{h\to 0} \sqrt[q]{1+h} &= 1 && (1)\\
\lim_{x\to 1} \sqrt[q]{x} &= 1 && (2)\\
\lim_{h\to 0} (1+h)^r &= 1 && (3)\\
\lim_{x\to 0} x^r &= 1 && (4)\\
\lim_{x\to 1} \frac{\sqrt[q]{x}-1}{x-1} &= 1/q && (5)
\end{aligned}$$

Reason: If $h > 0$ then

$$1 \leq \sqrt[q]{1+h} \leq (1+h)$$

so it follows from the Pinching Principle that

$$\lim_{h\to 0^+} \sqrt[q]{1+h} = 1.$$

On the other hand, if $-1 < h < 0$ then

$$1+h \leq \sqrt[q]{1+h} \leq 1,$$

so, again by applying the Pinching Principle,

$$\lim_{h\to 0^-} \sqrt[q]{1+h} = 1.$$

Therefore (1) is true. We already know that (1) and (2) are equivalent.

Next, since $x^r = (\sqrt[q]{x})^p$ for $x > 0$, (3) follows from the (1) and our theorem on limits of products, and (3) and (4) are equivalent.

Finally, first suppose that $x > 0$. Put $y = \sqrt[q]{x}$, so $x = y^q$. Then

$$\begin{aligned}
\frac{\sqrt[q]{x}-1}{x-1} &= \frac{y-1}{y^q-1}\\
&= \frac{1}{1+y+\cdots+y^{q-1}}\\
&= \frac{1}{1+x^{1/q}+x^{2/q}+\cdots+x^{(q-1)/q}}
\end{aligned}$$

Therefore

$$\lim_{x\to 1} \frac{\sqrt[q]{x}-1}{x-1} = \lim_{x\to 1} \frac{1}{1+x^{1/q}+x^{2/q}+\cdots+x^{(q-1)/q}} = \frac{1}{q}$$

since each of the q terms in the denominator has a limit of 1.

2 Exponential and Logarithm functions

We know that for $0 \le x < 1$ that

$$1 + x \le \exp(x) \le \frac{1}{1-x}.$$

Therefore for $0 \le x < 1$ that

$$1 - x \le \frac{1}{\exp(x)} \le \frac{1}{1+x}.$$

If we put $z = -x$ and recall that $\exp(x)\exp(-x) = 1$ then for $-1 < z \le 0$ we have

$$1 + z \le \exp(z) \le \frac{1}{1-z}.$$

We have shown

Lemma 1 *If* $|x| < 1$ *then*

$$1 + x \le \exp(x) \le \frac{1}{1-x}$$

Theorem 2

$$\begin{aligned}
\lim_{h\to 0} \exp(h) &= 1 \\
\lim_{h\to 0} \exp(a+h) &= \exp(a) \\
\lim_{h\to 0} \frac{\exp(h)-1}{h} &= 1 \\
\lim_{h\to 0} \frac{\exp(a+h)-\exp(a)}{h} &= \exp(a)
\end{aligned}$$

Reason: The key is the first assertion. We will use the Pinching Principle. We know from Lemma 1 that if $|h| < 1$ then

$$1 + h \le \exp(h) \le \frac{1}{1-h}.$$

Since the extreme expressions converge to 1 as h approaches 0, so does the middle expression.

The second formula follows from the first by observing that $\exp(a+h) = \exp(a)\exp(h)$.

For the third claim, we have to consider right and left hand limits. Observe that by subtracting 1 from each expression in Lemma 1 we have

$$h \le \exp(h) - 1 \le \frac{h}{1-h} \tag{6}$$

for $|h| < 1$. Therefore, if $0 < h < 1$ it follows from (6) that

$$1 \le \frac{\exp(h)-1}{h} \le \frac{1}{1-h}. \tag{7}$$

Applying the Pinching Principle to (7) gives

$$\lim_{h\to 0^+} \frac{\exp(h)-1}{h} = 1. \tag{8}$$

On the other hand, if $-1 < h < 0$ dividing (6) through by h gives

$$1 \geq \frac{\exp(h)-1}{h} \geq \frac{1}{1-h}. \tag{9}$$

Applying the Pinching Principle to (9) gives

$$\lim_{h\to 0^-} \frac{\exp(h)-1}{h} = 1. \tag{10}$$

Combining (8) and (10) tells us that

$$\lim_{h\to 0} \frac{\exp(h)-1}{h} = 1. \tag{11}$$

The last formula follows from (11) by remembering that

$$\exp(a+h) - \exp(a) = \exp(a)\left(\exp(h) - 1\right).$$

As for the natural logarithm function, recall Lemma 12. Since the natural logarithm function, ln is strictly increasing, the lefthand inequality in the Lemma tells us that $|x| < 1$ then

$$\ln(1+x) \leq x.$$

On the other hand, if we let $x = z/(1+z)$ and assume that $z > -1/2$, then $|x| < 1$, $1 - x = 1/(1+z)$, and the righthand inequality in the Lemma becomes

$$\exp\left(\frac{z}{1+z}\right) \leq 1 + z.$$

Taking the natural logarithm of both sides of this inequality gives

$$\frac{z}{1+z} \leq \ln(1+z).$$

Altogether this tell us that if $|h| < 1/2$ then

$$\frac{h}{1+h} \leq \ln(1+h) \leq h. \tag{12}$$

Straightaway we see from the Pinching Principle that

$$\lim_{h\to 0} \ln(1+h) = 0 = \ln(1)$$

so the natural logarithm function is continuous at 1. In fact, the natural logarithm function is continuous:

$$\begin{aligned}
\lim_{h\to 0} \ln(a+h) &= \lim_{h\to 0} \ln(a) + \ln(1 + (h/a)) \\
&= \lim_{k\to 0} \ln(a) + \ln(1+k)) \\
&= \ln(a),
\end{aligned}$$

where we have used the Shift and Scale theorem to replace h/a with k since $a > 0$. Remember that $a > 0$ since a is assumed to be in the domain of the natural logarithm function.

What is more, if $0 < h < 1/2$ we see that

$$\frac{1}{1+h} \le \frac{\ln(1+h)}{h} \le 1$$

so

$$\lim_{h\to 0^+} \frac{\ln(1+h)}{h} = 1.$$

On the other hand, if $-1/2 < h < 0$ we have

$$1 \le \frac{\ln(1+h)}{h} \le \frac{1}{1+h}$$

so

$$\lim_{h\to 0^-} \frac{\ln(1+h)}{h} = 1.$$

This tells us that

$$\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1,$$

so $\ln(1+h)/h$ has a removable discontinuity at $h = 0$.

3 Trigonometric Limits

We already know that

$$\lim_{h\to 0^+} \frac{\sin(x)}{x} = 1.$$

Since

$$\frac{\sin(-x)}{-x} = \frac{\sin(x)}{x}$$

we have

$$\lim_{h\to 0} \frac{\sin(x)}{x} = 1.$$

Next, since

$$0 \le 1 - \cos(x) \le 1 - \cos^2(x) = \sin^2(x) \le x^2$$

for $|x| \le \pi/2$ it follows from the Pinching Principle that

$$\lim_{x\to 0} 1 - \cos(x) = 0$$

which is equivalent to

$$\lim_{x\to 0} \cos(x) = 1.$$

We are now in a position to show that cosine is continuous:

$$\lim_{h\to 0} \cos(a+h) = \lim_{h\to 0} \cos(a)\cos(h) - \sin(a)\sin(h) = (\cos(a) \times 1) - (\sin(a) \times 0) = \cos(a)$$

A similar computation shows that sine is continuous.

In fact, since

$$\frac{1-\cos(x)}{x^2} = \frac{1-\cos^2(x)}{x^2(1+\cos(x))} = \left(\frac{\sin(x)}{x}\right)^2 \frac{1}{1+\cos(x)}$$

we have

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)^2 \frac{1}{1+\cos(x)} = 1^2 \times \frac{1}{2} = \frac{1}{2}$$

Now, by using the addition formula for cosine we can find the instantaneous rate of change of the cosine function at $x = a$. First, the average rate of change can be written as

$$\begin{aligned}\frac{\cos(a+h)-\cos(a)}{h} &= \frac{\cos(a)\cos(h)-\sin(a)\sin(h)-\cos(a)}{h}\\ &= -\sin(a)\frac{\sin(h)}{h} - \cos(a)h\frac{1-\cos(h)}{h^2}\end{aligned}$$

so taking the limit as h approaches 0 we get the instantaneous rate of change:

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(a+h)-\cos(a)}{h} &= \lim_{h\to 0} -\sin(a)\frac{\sin(h)}{h} - \cos(a)h\frac{1-\cos(h)}{h^2}\\ &= (\sin(a)\times 1) - \left(\cos(a)\times 0\times\frac{1}{2}\right)\\ &= -\sin(a)\end{aligned}$$

4 Absolute values

We already know that

$$\lim_{x\to a} f(x) = 0$$

is equivalent to

$$\lim_{x\to a} |f(x)| = 0.$$

This is not the case in general. For example, if $f(x) = 1$ for $x > 0$ and $f(x) = -1$ for $x < 0$, then f does not have a limit at 0, but $|f(x)|$ does. We have to settle for:

Theorem 3 *If*

$$\lim_{x\to a} f(x) = L$$

then

$$\lim_{x\to a} |f(x)| = |L|$$

Reason: We only need to worry about what happens when $L \neq 0$. Since $f(x) = L \times (f(x)/L)$ the Shift and Scale theorem tells us it is sufficient to consider the case where $L = 1$. In this case

$$0 \leq ||f(x)| - 1| \leq ||f(x)| - 1| \times ||f(x)| + 1| = \left|f(x)^2 - 1\right|.$$

Since

$$\lim_{x\to a} f(x) = 1$$

we know that
$$\lim_{x \to a} f(x)^2 = 1$$
which tells us that
$$\lim_{x \to a} \left| f(x)^2 - 1 \right| = 0.$$
It then follows from the Pinching Principle that
$$\lim_{x \to a} ||f(x)| - 1| = 0,$$
which is equivalent to
$$\lim_{x \to a} |f(x)| = 1.$$